ELLIPTIC CURVES IN TWO-DIMENSIONAL ABELIAN VARIETIES AND THE ALGEBRAIC INDEPENDENCE OF CERTAIN NUMBERS

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I. Introduction. A fruitful line of study in transcendental number theory has been an investigation into the number of algebraically independent values which belong to some prescribed set. The points under consideration are usually associated with the ordinary exponential function, or, more recently, with a Weierstrass elliptic function. In this paper we find some conditions which imply that a nontrivial one-parameter subgroup of a two-dimensional abelian variety is contained in an elliptic curve. From this we deduce several consequences concerning the algebraic independence (or transcendence) of certain values.

THEOREM. *Let A be a two-dimensional abelian variety defined over $\bar{\mathbf{Q}}$ and $\phi: \mathbf{C} \to A(\mathbf{C})$ a nontrivial analytic homomorphism which is defined over some subfield K of $\mathbf{C}$ (i.e., $\phi'(0) \in \mathfrak{Z}_A(K)$ where $\mathfrak{Z}_A$ denotes the tangent space of A at its identity element). Suppose y_0, y_1, y_2, y_3 are linearly independent complex numbers with $\phi(y_i) \in A(K)$ $(0 \le i \le 3)$ such that either*

(a) *$\phi(y_0) \in A(\bar{\mathbf{Q}})_{\text{tors}}$, or*

(b) *$\phi(y_0) \in A(\bar{\mathbf{Q}})$ and $y_0, y_1, y_2, y_3 \in K$.*

Then trans $\deg_{\mathbf{Q}} K \le 1$ *implies that $\overline{\phi(\mathbf{C})}$ is an elliptic curve.*

COROLLARY 1 (Elliptic analogue to the Brownawell–Waldschmidt theorem; [3], [10]). *Let $\wp(z)$ be a Weierstrass elliptic function with algebraic invariants and let $\mathcal{O}$ denote the ring of multiplications of $\wp$. Suppose that $\{u_1, u_2\}$ are $\mathcal{O}$-linearly independent and $\{v_1, \dots, v_4\}$ are $\mathbf{Z}$-linearly independent sets of complex numbers with $\wp(u_1 v_1)$ and $\wp(u_2 v_1)$ algebraic. If all of $\wp(u_i v_j)$ are defined, then at least two of*

$$u_i,\ v_j,\ \wp(u_i v_j) \quad (1 \le i \le 2,\ 1 \le j \le 4)$$

are algebraically independent.

Proof. Let E be the elliptic curve associated with $\wp(z)$, put $A = E \times E$ and

$$\phi(z) = \big(1, \wp(u_1 z), \wp'(u_1 z), 1, \wp(u_2 z), \wp'(u_2 z)\big).$$

$\phi(\mathbf{C})$ is Zariski dense in A since the $\mathcal{O}$-linear independence of u_1, u_2 implies that $\wp(u_1 z)$ and $\wp(u_2 z)$ are algebraically independent [4].

Put $K = \bar{\mathbf{Q}}(u_i, v_j, \wp(u_i v_j))$, $1 \le i \le 2$, $1 \le j \le 4$. Then $\phi'(0) \in \mathfrak{Z}_A(K)$, $\phi(v_1) \in A(\bar{\mathbf{Q}})$ and case (b) of the Theorem implies trans $\deg_{\mathbf{Q}} K \ge 2$. □

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