THE UNIQUENESS OF DECOMPOSITION OF A CLASS OF MULTIVALENT FUNCTIONS

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1. Introduction and statement of main theorem. Let P be a nonconstant polynomial. A curve ℓ will be called a *curved P-ray* if there is a path γ from $[0, \infty)$ onto ℓ such that $\gamma(t) \to \infty$ as $t \to \infty$, and $P \circ \gamma$ is one-to-one. Although a curved P-ray may contain critical points of P, we will be interested here in curved P-rays that do not.

Let $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{P}$ denote (respectively) the unit disc $\{z : |z| < 1\}$, the complex plane, and the Riemann sphere. Also, for any subset A of $\mathbf{C}$ let $\mathrm{Int}(A)$, $\mathrm{Bd}(A)$, $\mathrm{Ext}(A)$, and $\mathrm{Cl}(A)$ denote (respectively) the interior, boundary, exterior, and closure of A in $\mathbf{C}$. Denote by S the familiar class of all functions f analytic and univalent in $\mathbf{B}$ satisfying $f(0) = 0$ and $f'(0) = 1$.

Let f be a function which is analytic in $\mathbf{B}$ and has $p-1$ critical points (counting multiplicity). Also, suppose that $f = P \circ \phi$, where P is a polynomial of degree p and $\phi \in S$. This decomposition may not be unique in the sense that there may be another polynomial Q of degree p and another univalent function $\sigma \in S$ such that $f = Q \circ \sigma$. At the end of this paper we give an example of a function with non-unique decomposition. This example can be read independently of the rest of the paper. For another example see Lyzzaik [5].

The purpose of this paper is to give a quite general sufficient condition that guarantees unique decomposition, as follows.

THEOREM 1. *Let f be a function which is analytic in $\mathbf{B}$ and has $p-1$ critical points (counting multiplicity). Suppose $f = P \circ \phi$, where P is a polynomial of degree p and $\phi \in S$. Also, suppose that $B = P^{-1}\{P(z) : z$ is a critical point of $P\}$, and that there is a disjoint collection W of curved P-rays ℓ in $\mathbf{C} - \phi(\mathbf{B})$ such that $B \cap (\mathbf{C} - \phi(\mathbf{B})) \subset \bigcup_{\ell \in W} \ell$. If $f = Q \circ \psi$, where Q is a polynomial of degree p and $\psi \in S$, then Q and ψ are identical to P and ϕ, respectively.*

Note that B is the finite set of critical points of P and points mapped by P to the images of critical points. Since $\phi(\mathbf{B})$ contains all critical points of P it follows that $A = B \cap (\mathbf{C} - \phi(\mathbf{B}))$ is finite and contains no critical point of P.

The example we give at the end of this paper is one of the simplest functions that does not satisfy the hypotheses of the theorem.

We will show that the class of functions described in this theory properly contains the class $K(p)$ of close-to-convex functions of order p as defined by Livingston [3].

2. Proof of theorem. The proof of Theorem 1 will be executed in a sequence of lemmas. For convenience let $\Phi = \phi(\mathbf{B})$, and let $A = B \cap (\mathbf{C} - \Phi)$.

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