REPRESENTING HOMOLOGY CLASSES OF ALMOST DEFINITE 4-MANIFOLDS

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1. Introduction. In this note we wish to apply results of gauge theory (cf. [1], [2], [3]) to study which 2-dimensional homology classes are representable by embedded 2-spheres for almost definite 4-manifolds with odd intersection form. We work throughout in the differentiable category. Most of our results will concern simply connected 4-manifolds. Here results of Wall [15] and Freedman [5] imply that a simply connected 4-manifold with odd intersection form is homeomorphic to the connected sum of p copies of $\mathbf{C}P^2$ and q copies of $\overline{\mathbf{C}P}^2$, where $\overline{\mathbf{C}P}^2$ denotes $\mathbf{C}P^2$ with the other orientation. Let $M(p,q)$ denote a differentiable 4-manifold which is homeomorphic to $p\mathbf{C}P^2 \# q\overline{\mathbf{C}P}^2$. Note that Donaldson has shown that there exist examples of manifolds $M(1,9)$ which are not diffeomorphic to $\mathbf{C}P^2 \# 9\overline{\mathbf{C}P}^2$. Our results will concern almost definite manifolds where $p=1$ or 2.

Let us recall what Donaldson's Theorems A, B, and C say (cf. [1], [2]). Theorem A says that a definite simply connected 4-manifold must have standard intersection form. It was applied by Kuga [8] and Suciu [13] to study the problem of representing homology classes in $S^2 \times S^2$ and $\mathbf{C}P^2$ by embedded spheres and to give estimates on the number of double points of immersed spheres that represent the homology class. In the course of proving their results, they also gave results for when some rather special homology classes in manifolds $M(p,q)$ are represented by embedded spheres. An alternate proof of Kuga's theorem was given by Fintushel and Stern [3], and their techniques will be the basis of our results on $M(1,1)$ and $M(1,2)$. Theorem B says that a simply connected spin 4-manifold with $b_2^+=1$ must have intersection form a standard hyperbolic form of rank 2. Theorem C says that a simply connected spin 4-manifold with $b_2^+=2$ must have intersection form the direct sum of two copies of the standard hyperbolic form.

Our main results are the following.

THEOREM 1. *If x, y represent generators of $H_2(M(1,1))$ with $xy=0$, $x^2=-y^2=1$, then $ax+by$ is not represented by an embedded sphere if $||a|-|b||\geq 2$, except for $\pm(0,2)$, $\pm(2,0)$, $\pm(1,-1)$. If $M(1,1)$ is diffeomorphic to $\mathbf{C}P^2\#\overline{\mathbf{C}P}^2$, then $ax+by$ is represented by an embedded sphere if and only if $||a|-|b||\leq 1$ or $(a,b)=\pm(0,2)$ or $\pm(2,0)$ or $\pm(1,-1)$.*

THEOREM 2. *Let x be a characteristic homology class in $H_2(M(1,2))$.*
(i) *If x is represented by an embedded sphere, then $x^2=-1$.*
(ii) *If $M(1,2)$ is diffeomorphic to $\mathbf{C}P^2\#2\overline{\mathbf{C}P}^2$ and $x^2=-1$, then x is represented by an embedded sphere.*

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