INVARIANT PSEUDODIFFERENTIAL OPERATORS ON TWO STEP NILPOTENT LIE GROUPS, II

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In [7] a method was given for constructing parametrices and inverses for invariant hypoelliptic pseudodifferential operators which are homogeneous with respect to the natural dilations on a step two nilpotent Lie group. The construction made use of a calculus for invariant pseudodifferential operators described in [6]. It will be shown here that a similar calculus is also valid in the case of arbitrary dilations on a step two group. The parametrix construction of [7] can then be easily extended to include operators homogeneous with respect to arbitrary dilations. As noted in [8], this construction can be "microlocalized".

In [4] Melin gave a somewhat different parametrix construction on the Heisenberg group and extended this procedure to arbitrary graded Lie groups with the natural dilations in [5]. Glowacki's construction of a commutative approximate identity, given in [2] for arbitrary dilations on the Heisenberg group, makes use of the parametrix construction in [4]. A pseudodifferential operator calculus such as that given below is a prerequisite for extending the results of [2] and [4] to all step two groups with arbitrary dilations.

The classes of pseudodifferential operators considered here differ from those considered in [6] in that here we require estimates for derivatives in all directions, not just the orbit directions. The asymptotic formula (17) for a composition product $p\#q$ is also valid for the classes considered in [6], since the estimates for derivatives of $p\#q$ in the orbit directions will be seen to depend only on estimates for derivatives of p and q in the orbit directions.

The point to be made in this paper is that the calculus in the orbit directions follows naturally from the Weyl calculus of Hörmander [3], while the estimates in non-orbit directions can then be obtained by making use of identities derived from the Lie algebra structure. We note that the development here is somewhat more natural than that in [6], since we have not needed to polarize the orbits.

DEFINITION. A family of dilations on a finite-dimensional Lie algebra $\mathcal{G}$ is a one-parameter family $\delta=\{\delta_r : r>0\}$ of automorphisms of $\mathcal{G}$ such that

$$\delta_r e_j = r^{\mu_j} e_j, \quad \mu_j > 0, \tag{1}$$

for some basis $\{e_1,\dots,e_n\}$ for $\mathcal{G}$. A connected, simply connected nilpotent Lie group is said to be a homogeneous group if its Lie algebra is endowed with a family of dilations ([1]).

Without loss of generality we may assume that $\min \mu_j = 1$. It can be easily shown that there is a linearly independent set $S=\{e_1,\dots,e_N\}$ which generates $\mathcal{G}$, satisfies (1), and such that $\mathcal{G}_1=\operatorname{span} S$ intersects $\mathcal{G}_2=[\mathcal{G},\mathcal{G}]$ trivially. Assuming for the

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