THE BEST CONSTANT IN A BMO-INEQUALITY FOR THE BEURLING–AHLFORS TRANSFORM

T. Iwaniec

Introduction. The Beurling–Ahlfors operator Tf for $f \in L^2(\mathbf{C})$ is defined by the following relation between Fourier transforms:

$$(Tf)^\wedge(\xi) = m(\xi)\hat{f}(\xi),$$

where $m(\xi) = (\xi/|\xi|)^2$ for every $\xi \in \mathbf{C} - \{0\}$.

Clearly T is a unitary operator on $L^2(\mathbf{C})$ commuting with translations and dilations. Another definition of T is the convolution formula

$$(Tf)(z) = -\text{p.v.}\, \frac{1}{\pi} \int \frac{f(t)\, d\sigma(t)}{(z-t)^2},$$

where p.v. means the principal value and $d\sigma(t)$ the Lebesgue measure in $\mathbf{C}$. This operator can be regarded as an analogue of the Hilbert transform in the complex plane with an even kernel.

The importance of the Beurling–Ahlfors operator to the elliptic equations ([3], [4], [13]) as well as to quasiconformal mappings in the plane lies in the fact that it changes the complex derivative $\partial_{\bar{z}}$ into ∂_z: in symbols,

$$T\left(\frac{\partial w}{\partial \bar{z}}\right) = \frac{\partial w}{\partial z} \tag{1}$$

for every w in the Sobolev space $\mathcal{W}_2^1(\mathbf{C})$. We shall appeal to this formula to evaluate Tf for some particular functions f.

As an operator of Calderon–Zygmund type, the Beurling–Ahlfors transform is bounded in $L^p(\mathbf{C})$ for all $1 < p < \infty$. This breaks down for $p = \infty$. However, in this limiting case T extends to a bounded operator from $L^\infty(\mathbf{C})$ into BMO-spaces [12].

Fix $1 \le p < \infty$. A function $f \in L^p_{\text{loc}}(\mathbf{R}^n)$ is said to be of bounded mean oscillation (briefly, BMO_p) if

$$\|f\|_{\text{BMO}_p} = \sup_B \left(\fint_B \left| f(x) - \fint_B f(y)\, dy \right|^p dx \right)^{1/p} < \infty,$$

where the supremum is taken over all balls B in $\mathbf{R}^n$ and

$$\fint_B f(y)\, dy = \frac{1}{|B|} \int_B f(y)\, dy = f_B$$

is the average of f on B.

Received June 4, 1985. Revision received September 4, 1985.

This research was done while the author was visiting the University of Texas at Austin in 1984–1985.

Michigan Math. J. 33 (1986).