ON THE MEAN VALUE THEOREM FOR ANALYTIC FUNCTIONS

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In all that follows $D(r, a)$ will denote the disk $|z-a|<r$ in the complex plane C; we abbreviate $D(r, 0)$ by $D(r)$ and $D(1)$ by D. The plane including the point at infinity will be denoted by C_∞. For any region $R \subset C$, $B(R)$ will denote the family of all analytic functions f on D for which $f'(D) \subset R$. The mean value theorem of the differential calculus can be viewed as saying that if f is a differentiable real valued function on an open interval I for which $f'(I) \subset J$, then $f[b,a]=(f(b)-f(a))/(b-a) \in J$ for all pairs of distinct points a, b in I. The direct analogue of this is not true for analytic functions in D since, for example, $f[b,a]$ can be 0 even though $f'(D)$ does not contain 0. As a way of quantifying the degree to which the mean value theorem does hold for functions in $B(R)$, one is led to consider the supremum, denoted by $m(R)$, of all $r<1$ for which $f \in B(R)$ implies that $f[b,a] \in R$ for all $a, b \in D(r)$. (Here and in what follows, $f[b,a]$ is defined to be $f'(b)$ in the case that $a=b$.)

In the course of this discussion use is made of several theorems of classical complex analysis such as Bloch's theorem and the uniformization theorem; since these facts are well known and fully treated in many texts, no explicit references are given for them. The main facts established in what follows appear as indented statements, labeled with the letters (A)–(G).

1. Conditions for $m(R)<1$ and $m(R)>0$. The purpose of this section is to characterize those $R \subset C$ for which the study of $m(R)$ is interesting, that is, to determine when $0<m(R)<1$. We begin by pointing out that

(A) $m(R)=1$ if and only if R is convex.

The sufficiency of the convexity is well known and is easy to see since $f[b,a]=\int_0^1 f'(a+t(b-a))\,dt$. Conversely, let $R \subset C$ be nonconvex. Then ([2, Theorem 4.8]) there exists $c \in \partial R$ such that for some $r>0$ and real t, $e^{it}(R-c)$ contains the set $S=\{z: 0<|z|<r, \operatorname{Re}\{z\} \geq 0\}$. From this it follows that for some $\delta \in (0, r/2)$, $e^{it}(R-c)$ contains the set $T=S \cup D(\delta, ir/2) \cup D(\delta, -ir/2)$, which is symmetric with respect to the real axis. Let g' map D one-to-one onto T in such a way that g' is real on the real axis and $g'(-e^{i\theta})=-ir$ and $g'(-e^{-i\theta})=ir$. Then for $\theta>0$ sufficiently small we have $g[-e^{-i\theta}, -e^{i\theta}]<0$. But for s real, $g[-se^{-i\theta}, -se^{i\theta}]$ is real and tends to $g'(0)>0$ as s tends to 0. Thus there exist distinct points a, b in D such that $g(a)=g(b)$. But then $f=e^{-it}g+cz \in B(R)$ and $f[b,a]=c \notin R$, so that $m(R)<1$. It is to be noted that for any given nonconvex R a construction along these lines may be used to determine an upper bound (less than 1) for $m(R)$.

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