PERTURBATIONS OF MATRIX ALGEBRAS

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Two operator algebras acting on the same Hilbert space are said to be close if their unit balls are close in the Hausdorff metric. We are interested in algebraic and spatial characteristics of an algebra which persist under small perturbations. Even in low dimensions there is much pathology, and several examples are given to demonstrate this. On the other hand, certain classes of algebras behave very well. Two such classes are considered here: semi-simple algebras and reflexive algebras with distributive lattices. In these two cases, any algebra sufficiently close to one of these algebras is similar to it via an operator close to the identity.

The study of perturbations of operator algebras was initiated by Kadison and Kastler [12] for von Neumann algebras. This has stimulated a lot of further work in W^* and C^* algebras [3, 4, 5, 6, 7, 11, 17, 18, 19]. There has also been some work on nonself-adjoint algebras, notably Lance's work [14] on nest algebras. There is a strong connection between perturbation results and classification of algebras up to similarity. This comes out strongly in [15, 16] and [9] where the similarity theory for nest algebras is obtained. Some examples related to ours are obtained in [13]. More generally still, various authors have considered perturbations of arbitrary Banach algebras [10, 20].

In the self-adjoint case, it often turns out that close algebras are unitarily equivalent via a unitary close to the identity. In [12], it was shown that close von Neumann algebras can be decomposed into summands of various types, preserving closeness. Then in [4, 17] it was shown that close type I von Neumann algebras are unitarily equivalent. No counterexample to "close implies unitarily equivalent" is known among von Neumann algebras or separable C^* algebras. However, there is a counterexample [3] among larger C^*-algebras. Furthermore, there is some strange behaviour known about C^* algebras almost contained in others [7, 11].

For nests, it is not possible to use unitaries to get all close algebras. However, Lance [14] has shown that close nests yield close algebras via a similarity close to the identity. One of the motivations of this paper was an attempt to generalize the result to a larger class of nonself-adjoint algebras—the reflexive algebras with commutative subspace lattices. The results in this paper deal only with the finite-dimensional case, although the remarks of Sections 4 and 5 indicate some of the possibilities in the infinite-dimensional case. These ideas will be pursued elsewhere [22].

1. Preliminaries. In this paper, $\mathcal{H}$ will always be a finite-dimensional Hilbert space. The algebra of $n \times n$ matrices is denoted by $\mathfrak{M}_n$, and $\mathcal{L}(\mathcal{H})$ denotes this

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