ON CONTACT BETWEEN SUBMANIFOLDS

James A. Montaldi

The principal aim of this article is to clarify the relationship between the contact of submanifolds and the singularity type (more precisely, the $\mathcal{K}$-class) of maps. In [3], Golubitsky and Guillemin consider the equidimensional case, but the general case has not been treated previously.

We define the local notion of contact type in the obvious way: Given two pairs of submanifold-germs at the origin in $\mathbf{R}^n$, then the pairs have the same contact type if there is a diffeomorphism-germ of $(\mathbf{R}^n, 0)$ taking one pair to the other. (Clearly, the dimensions of one pair must be the same as the dimensions of the other.) We denote the contact type of X and Y by $K(X, Y)$.

The main result is the following.

THEOREM. *For* $i=1,2$, *let* $g_i\colon (X_i, x_i) \hookrightarrow (\mathbf{R}^n, 0)$ *be immersion-germs and* $f_i\colon (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ *be submersion-germs, with* $Y_i = f_i^{-1}(0)$. *Then the pairs* (X_1, Y_1) *and* (X_2, Y_2) *have the same contact type if and only if* $f_1 \circ g_1$ *and* $f_2 \circ g_2$ *are* $\mathcal{K}$*-equivalent.*

This relationship also serves to elucidate the work of Porteous and others [6, 7] on the distance-squared function and the geometry of submanifolds of Euclidean space. The singularities of the distance-squared functions that occur for a given submanifold can be viewed as types of contact of the submanifolds with hyperspheres in the Euclidean space. In subsequent articles we will take this idea further and consider the contact with spheres of higher codimension.

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Since completing this work I have learned that J. W. Bruce has thought about similar questions. In particular, the Symmetry Lemma is proved in an article of his entitled "Wavefronts and Parallels in Euclidean Space," to appear in the Mathematical Proceedings of the Cambridge Philosophical Society.

Preliminaries. Recall that two map-germs $\phi, \psi\colon (\mathbf{R}^k, 0) \to (\mathbf{R}^p, 0)$ are $\mathcal{K}$-equivalent if there are diffeomorphism-germs h of $(\mathbf{R}^k, 0)$ and H of $(\mathbf{R}^k \times \mathbf{R}^p, (0,0))$ such that $H(x, 0) = (h(x), 0)$ and $H(x, \phi(x)) = (h(x), \psi \circ h(x))$. Equivalently, we require H to map the graph of the zero-map to itself and the graph of ϕ to the graph of ψ. Also useful is the fact that H can be written in the form $H(x, y) = (h(x), \theta(x, y))$, where $\theta\colon (\mathbf{R}^k \times \mathbf{R}^p, (0,0)) \to (\mathbf{R}^p, 0)$ and the differential of θ with respect to its second argument $d_y\theta$ at 0 is surjective.

One other fact we use is the following: Let $\phi, \psi\colon (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ be map-germs, and $\bar{\phi}, \bar{\psi}\colon (\mathbf{R}^{n+k}, 0) \to (\mathbf{R}^{p+k}, 0)$ be suspensions of ϕ and ψ respectively.

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