CELLULAR-INDECOMPOSABLE OPERATORS AND BEURLING'S THEOREM

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1. Introduction. Let H be a Hilbert space with norm $\| \ \|$ consisting of functions analytic on the open unit disk Δ. We will assume that H has the following properties.

(1) The polynomials are dense in H.

(2) Multiplication by z, T_z, is a bounded linear operator on H.

(3) If $zg \in H$ for some function g analytic on Δ, then $g \in H$.

(4) For each point $b \in \Delta$, the linear functional of evaluation at b, λ_b, is continuous with respect to the norm of H.

Requiring that H have property (3) is actually equivalent to requiring that T_z be bounded below (see Proposition 2). In this paper we will be concerned primarily with the operator T_z on H.

For $-\infty < \alpha < \infty$, let D_α represent the Hilbert space of analytic functions f on Δ satisfying $\|f\|_\alpha < \infty$, where if $f = \sum_{n=0}^{\infty} a_n z^n$, $\|f\|_\alpha^2 = \sum_{n=0}^{\infty} (n+1)^\alpha |a_n|^2$. It is not difficult to verify that the spaces D_α with norm $\| \ \|_\alpha$ satisfy properties (1)–(4) above. Note that D_{-1}, D_0, and D_1 are the Bergman, Hardy, and Dirichlet spaces respectively. Also note that the operator T_z on D_α corresponds to the unilateral weighted shift with weight sequence

$$\left\{\left(\frac{n+2}{n+1}\right)^{\alpha/2}\right\}_{n=0}^{\infty}$$

relative to the orthonormal basis

$$\left\{\left(\frac{1}{n+1}\right)^{\alpha/2} z^n\right\}_{n=0}^{\infty} \text{ of } D_\alpha$$

(cf. [9]).

We say that a closed subspace $M \subset H$ is invariant if it is invariant under the operator T_z; that is, a subspace M is invariant if it is closed and $zM \subset M$. For a function $f \in H$, define $[f] = H$-closure of $\{pf : p \text{ is a polynomial}\}$. We say that an invariant subspace M of H is cyclic provided there is some function $f \in M$ such that $M = [f]$. We denote by $N_1 \ominus N_2$ the orthogonal complement of N_2 in N_1 for closed subspaces $N_2 \subset N_1 \subset H$; and by $M_1 \vee M_2$, the closed linear span of the subspaces M_1 and M_2 of H. For M_1 and M_2 invariant subspaces of H, observe that $M_1 \cap M_2$ and $M_1 \vee M_2$ are invariant.

Adopting terminology established in [8], we say that the operator T_z on H is cellular-indecomposable if $M_1 \cap M_2 \neq \{0\}$ for any two nonzero invariant subspaces M_1 and M_2 of H. The following proposition applies to T_z on D_0 since each function in the Hardy space is the quotient of H^∞ functions, and since for ϕ a multiplier of D_α and $f \in D_\alpha$, $\phi[f] \subset [f]$ (cf. [3, Proposition 7]).

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