ONE-SIDED CLOSED GEODESICS ON SURFACES

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Let M^2 be a closed Riemannian 2-manifold and let α be a non-trivial element of $\pi_1(M)$. Among the set of all smooth loops in M which are freely homotopic to a curve representing α, there is a shortest member $f: S^1 \to M$, which is a smooth closed geodesic. Both f and the image of f will not be unique, in general. If α is orientation-preserving, then it was shown in [2] that f has the least possible number of self-intersections, unless f factors through a covering. In particular, if α is represented by an embedded loop, then f is either an embedding or a double cover of an embedded one-sided curve.

If α is orientation-reversing, then any loop which is freely homotopic to a curve representing α is one-sided. The features of one-sided loops differ significantly from two-sided curves, in particular those properties associated with coverings. Thus covers of one-sided shortest geodesics are not necessarily shortest, unlike the two-sided case.

A specific example of the difficulties encountered in the one-sided situation is seen by starting with a flat Möbius band M^2 and putting a bump in it as in Figure 1.

The bump is formed by multiplying the metric by a rotationally symmetric function on the shaded disk in Figure 1. A large enough bump will force the shortest geodesic representing a generator α of $\pi_1(M^2)$ to go around the bump. It is now clear that a shortest geodesic representing α^2 will not double cover a shortest loop representing α. This contrasts with Lemma 1.3 of [2]. Note that there are at least two distinct shortest geodesics representing α, by the symmetry of the construction. One goes above and one below the bump.

Nonetheless, we will show in this paper that shortest one-sided geodesics still minimize the number of double points in their intersection sets.

DEFINITION. We say that a loop $f: S^1 \to M$ represents $\alpha \in \pi_1(M, x)$ if f is freely homotopic to a loop at x in the homotopy class (rel x) of α. $f \sim \alpha$ will be used to denote that f represents α.

DEFINITION. $f: \mathbf{R} \to M$ is length-minimizing (or shortest) if f is shortest on any compact arc $I \subset \mathbf{R}$, in the homotopy class relative to ∂I of f restricted to I.

DEFINITION. Two maps $f: \mathbf{R} \to M$ and $g: \mathbf{R} \to M$ are homotopic by a homotopy with compact support if there is a homotopy $H: \mathbf{R} \times I \to M$ with $H(s,0) = f(s)$, $H(s,1) = g(s)$ and if there is a $K > 0$ such that $|s| > K$ implies $H(s,0) = H(s,t)$ for all $0 \le t \le 1$. Equivalently, the homotopy only moves a compact arc of $\mathbf{R}$.

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