BRANCHED IMMERSIONS OF SURFACES

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Introduction. Let M and N be surfaces and $f: \partial M \to N$ an immersion. When can f be extended to an immersion or polymersion $F: M \to N$? (That is, locally $F(z) = z^n$, $n > 0$.) In this paper, we consider the case where M is a compact connected surface of genus k with n boundary components and $N = \mathbf{R}^2$. In addition, f will be required to be a *normal immersion,* that is, it has finitely many intersections and these are transverse double points. We also determine how many different extensions there are.

DEFINITION. Let $f: \partial M \to N$ be an immersion and let $F_1, F_2: M \to N$ be two extensions of f. F_1 and F_2 are *equivalent* if there is a diffeomorphism $h: M \to M$ such that $h|_{\partial M} = \mathrm{Id}$ and $F_1 = F_2 \circ h$.

C. Titus [8] solved the existence problem for $F: D^2 \to \mathbf{R}^2$. S. Blank [3] gave a different and elegant solution to both the existence and equivalence problems by associating to an immersion a natural 'word.' He showed that equivalence classes of immersions which extend f are in one-to-one correspondence with 'groupings' of the word. (These groupings are combinatorial structures on the word.) Blank's method will be generalized in this paper.

Later, K. Bailey [1], M. Barall [2], C. Ezell [4], G. Francis [5], M. Marx [7], and S. Troyer [9] modified some of Blank's techniques to deal with various other cases. The algorithms given in this paper require only the same concise conditions that were present in the simplest case.

ROTATION NUMBER OF f. Let $f: S^1 \to \mathbf{R}^2$. The rotation number (or tangent winding number) of f, $R(f)$, is the index of the map

$$x \mapsto \frac{f_x'(1)}{|f_x'(1)|} : S^1 \to S^1.$$

Given $f: \partial M \to \mathbf{R}^2$, we have $f = f_1 \cup \cdots \cup f_n$, where $f_i: S^1 \to \mathbf{R}^2$ is f restricted to the ith boundary component. Define the *total rotation number* of f to be $\tau(f) = R(f_1) + \cdots + R(f_n)$. By a result of A. Haefliger [6], if the compact surface M is immersed in the plane then $\tau(f) = X(M) = 2 - n - 2k$. However, this condition is not sufficient for f to extend, as can be seen from the example of Figure 1. This and subsequent figures represent images in $\mathbf{R}^2$.

THE BLANK WORD. Let $P_1, \ldots, P_m$ be the bounded connected components of $\mathbf{R}^2 - \mathrm{Im}(f)$. Choose $p_i \in P_i$. For each p_i construct a ray $a_i: [0, \infty) \to \mathbf{R}^2$ which is an embedding such that:

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