HOOLEY'S Δ_r-FUNCTIONS WHEN r IS LARGE

R. R. Hall

1. Introduction. In an important paper [3], Hooley introduced the function

$$\Delta_r(n) := \max_{u_1,\dots,u_{r-1}} \operatorname{card}\{d_1,\dots,d_{r-1} : d_1 \dots d_{r-1} \mid n, u_i < d_i \le eu_i \ \forall i\}.$$

I am interested here in upper bounds for the sum

$$S_r(x,y) := \sum_{n\le x} \Delta_r(n) y^{\omega(n)}, \quad y>0,$$

where $\omega(n)$ denotes the number of distinct prime factors of n. I follow Hall and Tenenbaum [2] in denoting by $\alpha(r,y)$ the infimum of the numbers ξ for which $S_r(x,y) \ll_\xi x(\log x)^\xi$ and in setting $A_r := \alpha(r,1)$. The function $\alpha(r,y)$ is known precisely, for $y \in \mathbf{R}^+ \setminus (\frac{1}{2},2)$ and in Theorem 1 I shorten the excluded interval. In Theorem 2 I improve the known upper bounds for A_r for $r \ge 4$: in particular $A_r < \sqrt{r} - 1$ $(4 \le r \le 18)$, and at least in this range, Theorem 1B [3] is reinstated (cf. [2] concerning this theorem). For $r \ge 19$, the result is better than $A_r < 33(r+7)/244$.

The applications of Hooley's "new technique" set out in [3] required upper estimates for A_r only. Not only does the more general function $\alpha(r,y)$ seem interesting, particularly since for certain y there is a simple formula for it, but in the cases $r=2,3,4$ the upper best bounds, viz

$$A_2 < .21969, \quad A_3 < .55153, \quad A_4 < .92752,$$

have depended on estimates for $\alpha(r,y)$, $y \ne 1$ [2]. Such information can be applied to the study of A_r in two ways: by virtue of the fact that $\alpha(r,y)$ is a convex function of $\log y$, (by Hölder's inequality applied to $S_r(x,y)$), and through the Iteration Inequality of Hall and Tenenbaum [2]: for $r \ge s \ge 1$, $y,z>0$,

$$2\alpha(r,y) \le \alpha(r,z) + (s-1)\max(sz-1,0) + \alpha(r-s+1, y^2/z)$$

$(\alpha(1,y) := y-1)$. The following information is available from [2]:

THEOREM A. *We have* $A_r \le r/4$, $(r \ge 5)$.

THEOREM B. *For* $r \ge 2$ *and* $y \notin (\frac{1}{2},2)$ *we have* $\alpha(r,y) = y-1+(r-1)\chi(y-1)$, *where* $\chi(u) := \max(0,u)$. *Moreover* $\alpha(r,y) \ge y-1+(r-1)\chi(y-1)$ *for all* y.

The second part of Theorem B follows directly from

$$\Delta_r(n) \gg 1 + \tau_r(n)/(\log n)^{r-1}.$$

The first part suggests the definitions:

Received May 25, 1984. Revision received January 14, 1985.
Michigan Math. J. 33 (1986).