REMARKS ON THE APPROXIMATION TO AN ALGEBRAIC NUMBER BY ALGEBRAIC NUMBERS

E. Bombieri and J. Mueller

1. Introduction. Let α be a real algebraic number and let k be a real algebraic number field, $\alpha \notin k$. The celebrated theorem of K. F. Roth ([12], [13]) asserts that α cannot be approximated too well by elements of k and, more precisely, for every $\epsilon > 0$ there is $c(\alpha, \epsilon) > 0$ such that for every $\beta \in k$

$$(1) \qquad |\alpha - \beta| > c(\alpha, \epsilon) H_k(\beta)^{-2-\epsilon},$$

where $H_k(\beta)$ is the field height, that is the largest coefficient of the integral polynomial with roots $\sigma(\beta)$, counted with multiplicity, for all distinct embeddings $\sigma: k \to \mathbf{C}$. Since it is known (see e.g. Schmidt [14, Ch. VIII, Th. 2A]) that there are infinitely many β's in the field k such that

$$(2) \qquad |\alpha - \beta| < c(\alpha) H_k(\beta)^{-2},$$

the above result of Roth is clearly best possible.

It is a well-known feature of Roth's theorem that inequality (1) is ineffective, in the sense that the proof yields the existence of the constant $c(\alpha, \epsilon)$ in (1) but does not allow the calculation of a lower bound for it. If we ask for effective lower bounds for $|\alpha - \beta|$ then our knowledge about approximations is much weaker than that given by (1). Let us define $\mu_{\text{eff}}(\alpha, k)$ to be the infimum of all μ's for which an inequality $|\alpha - \beta| > c(\alpha) H_k(\beta)^{-\mu}$ holds for every $\beta \in k$ and some effectively computable $c(\alpha) > 0$.

The first general improvement on the elementary bound $\mu_{\text{eff}}(\alpha, k) \leqq [k(\alpha): k]$ was obtained by Baker [1] using his theory of linear forms in logarithms, and eventually Feldman [11] proved (at least in the case $k = \mathbf{Q}$) that $\mu_{\text{eff}}(\alpha, k) \leqq \deg \alpha - \eta$, where $\eta = \eta(\alpha, k) > 0$ is a positive very small constant. For an account of this theory see Baker's monograph [2].

The Baker–Feldman theorem is the only non-trivial effective result available today valid for every algebraic number α and every number field k. On the other hand, for special numbers α better effective results are known, in particular: $\alpha = \xi^{1/r}$ with $\xi \in k$ ([3], [4], [6], [7]); α a cubic number ([7], [10]); some special algebraic numbers, such as $\alpha^r + m\alpha - 1 = 0$ ([5]); the typical situation here is the case in which $k = \mathbf{Q}$, while α is restricted in various ways.

In this paper we show that for any given α one can find algebraic number fields k for which precise information about effective approximation can be obtained.

THEOREM 1. *Let α be a real algebraic number of degree $r \geqq 3$ and let $\eta > 0$ be any positive constant. Then one can find infinitely many real algebraic number fields k of degree $r-1$ such that $\mu_{\text{eff}}(\alpha, k) \leqq 2 + \eta$.*

Received May 4, 1984. Final revision received April 22, 1985.
Michigan Math. J. 33 (1986).