SECONDARY INVARIANTS OF TRANSVERSELY HOMOGENEOUS FOLIATIONS

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1. Introduction. Let N be a smooth manifold and G a group of diffeomorphisms of N. A transverse (G, N) foliation F on a smooth manifold M consists of an open cover $\{U_\alpha\}_{\alpha \in A}$ of M together with submersions $\varphi_\alpha: U_\alpha \to N$ such that for each pair U_α, U_β with $U_\alpha \cap U_\beta \neq \varnothing$, there is an element $g_{\alpha\beta}$ of G with $g_{\alpha\beta} \circ \varphi_\beta = \varphi_\alpha$. If the dimension of N is q, then F is a smooth codimension q foliation on M and there is a well-defined map $\alpha_F: H^*(WO_q) \to H^*(M)$ giving the characteristic classes of the foliation. Let SL_{q+1} act on S^q as the space of directed lines in $\mathbf{R}^{q+1}$. In this note we prove the following.

THEOREM. *Let F be a transverse (SL_{q+1}, S^q) foliation on M and suppose that $\pi_1(M)$ is finitely generated. Let γ be an element of $H^*(WO_q)$. Then the set of values that $\alpha_F(\gamma)$ can take on is a finite subset of $H^*(M)$.*

This theorem is a generalization of the same result for transverse (PSL_2, S^1) foliations and the Godbillon–Vey invariant $h_1 c_1 \in H^*(WO_1)$ due to Brooks–Goldman. The proof appears in [3] which was the inspiration for this paper.

2. Characteristic classes for foliations. The material in this section is fairly standard so we shall recall it only briefly. For more details see [1], [2], and [11].

The differential graded algebra WO_q is

$$WO_q = R_q[c_1, \ldots, c_q] \otimes \Lambda(h_1, h_3, \ldots, h_{2s+1})$$

where the degree $c_i = 2i$, degree $h_i = 2i-1$, and $2s+1 = q$ or $q-1$. $R_q[c_1, \ldots, c_q]$ is the polynomial algebra truncated above degree $2q$ and $\Lambda(h_1, \ldots, h_{2s+1})$ is the exterior algebra on the h_i's. The DGA W_q is

$$W_q = R_q[c_1, \ldots, c_q] \otimes \Lambda(h_1, \ldots, h_q).$$

In both cases the differential is given by setting $d(c_i) = 0$, $d(h_i) = c_i$ and extending as a derivation.

Let F be a codimension q foliation on M and let $\tau \subset TM$ be its tangent bundle, where TM is the tangent bundle of M. The normal bundle of F is $\nu = TM/\tau$ and we write $\rho: TM \to \nu$ for the projection. A basic connection θ^b on ν is one whose covariant derivative ∇ satisfies $\nabla_X \rho(Y) = \rho([X, Y])$ for all $X \in \tau$.

Denote the space of differential forms on M by $A^*(M)$. Let θ^r be a Riemannian connection on ν. The map $\alpha_F: WO_q \to A^*(M)$ is given as follows: Let Ω^b be the curvature of θ^b and Ω^t the curvature of the connection $\theta^t = t\theta^b + (1-t)\theta^r$ interpolating between θ^b and θ^r. Then

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