THE REDUCED MINIMUM MODULUS

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Introduction. Let T be a bounded linear operator acting in a Banach space. The reduced minimum modulus of T will be defined by the equation

$$\gamma(T)=\begin{cases}\inf\{\|Tx\|:\operatorname{dist}(x,\ker T)=1\} & \text{if } T\neq 0,\\ 0 & \text{if } T=0.\end{cases}$$

The definition of $\gamma(T)$ is taken from [9, Ch. IV, §5] for $T\neq 0$. (If $T=0$ we put $\gamma(T)=0$, whereas in [9], $\gamma(T)=\infty$). Thus $\gamma(T)>0$ if and only if T has closed non-zero range. If T is invertible then $\gamma(T)=\|T^{-1}\|^{-1}$ and this shows that the function $T\to\gamma(T)$ is not continuous but it could have good local continuity properties.

In general $\lim_{n\to\infty}\gamma(T^n)^{1/n}$ does not exist and it is not known what conditions on T are equivalent to the existence of $\lim_{n\to\infty}\gamma(T^n)^{1/n}$. We mention the following known cases when $\lim_{n\to\infty}\gamma(T^n)^{1/n}$ exists:

(1) if T is Fredholm, $\lim_{n\to\infty}\gamma(T^n)^{1/n}$ is the radius of the largest open disk centered at 0, included in the Fredholm domain of T, such that $\dim\ker(T-\lambda)=\text{const.}$ for $\lambda\neq 0$ in the disk ([8]);

(2) if T is surjective or bounded from below, $\lim_{n\to\infty}\gamma(T^n)^{1/n}$ is the radius of the largest open disk centered at 0 such that $T-\lambda$ is surjective or bounded from below for λ in the disk ([10], [11]).

In this paper we investigate the properties of the reduced minimum modulus of operators acting in Hilbert spaces. In Section 1 we develop some general properties of $\gamma(T)$ and a related matrix representation of T (Theorem 1.5). Section 2 will be devoted to the study of the continuity properties of the function $\lambda\to\gamma(T-\lambda)$, $\lambda\in C$. The discontinuities of this function form a countable set and $\lim_{\lambda\to\mu}\gamma(T-\lambda)$ always exists. The set

$$\sigma_\gamma(T)=\left\{\mu\in C:\lim_{\lambda\to\mu}\gamma(T-\lambda)=0\right\}$$

is closed, non-empty, and obeys the spectral mapping theorem (Theorem 2.7). As seen in Theorem 2.5 and Proposition 2.6, $\rho_\gamma(T)$, the complement of $\sigma_\gamma(T)$, is the minimal open set where $T-\lambda$ has an analytic generalized inverse.

Section 3 deals with the problem of the existence of $\lim_{n\to\infty}\gamma(T^n)^{1/n}$ and the role $\sigma_\gamma(T)$ plays in this problem. A positive new result and a direct generalization of the result of [8] is the existence of $\lim_{n\to\infty}\gamma(T^n)^{1/n}$ for semi-Fredholm operators (see Remark after Corollary 3.4).

The last part of the paper, Section 4, contains some results on the effect of a compact perturbation K on $\sigma_\gamma(T+K)$ (see Theorem 4.4).

1. Preliminaries. Throughout the paper we shall denote by H a fixed complex Hilbert space, $H\neq\{0\}$, and T will be a fixed bounded linear operator acting in H.

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