THE EXTREMAL PSH FOR THE COMPLEMENT OF CONVEX, SYMMETRIC SUBSETS OF $\mathbf{R}^N$

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Introduction. For a compact subset E of $\mathbf{C}^N$, we define the extremal function

$$\Phi_E(z) = \sup\{|p(z)|^{1/\deg p}\}.$$

The supremum is taken over all polynomials in N complex variables with $\|P\|_E \leq 1$.

This function has been investigated and used in connection with polynomial approximation in $\mathbf{C}^N$ by Siciak [4]. Zahariuta [6] and with a different method Siciak [5] have shown that

$$(0.1) \qquad \log \Phi_E(z) = \sup\{u(z)\};$$

here the supremum is taken over all plurisubharmonic functions v with $v(z) \leq \log(|z|+1)+C$ and $v(z) \leq 0$ on E.

Similar functions have been used in connection with potential theory in $\mathbf{C}^N$ and also when investigating the complex Monge–Ampére equation; see for example Bedford [1, 2] and Bedford and Taylor [3] where further references can be found.

In this note we give a fairly explicit formula for Φ_E when E is convex, symmetric with respect to 0 and contained in $\mathbf{R}^N \subseteq \mathbf{C}^N$. This calculation will also give us a complex foliation of $\mathbf{C}^N$ such that Φ_E is harmonic on each leaf except at the intersections between the leaves and E.

Definitions and formulation of the main result. We will always consider $\mathbf{R}^N$ as a subset of $\mathbf{C}^N$. When we talk about the topology of a subset of $\mathbf{R}^N$ we usually refer to the $\mathbf{R}^N$-topology.

Let S_N denote the unit sphere in $\mathbf{R}^N$, and for $\xi \in S_N$, $z \in \mathbf{C}^N$ define $\xi \cdot z = \xi_1 z_1 + \cdots + \xi_N z_N$.

For a convex symmetric set $E \subseteq \mathbf{R}^N$ with nonempty interior (symmetric means symmetric with respect to 0, i.e. $E = -E$), we have a representation

$$(1.1) \qquad E = \{z \in \mathbf{C}^N : a(\xi)\xi \cdot z \in [-1, 1] \text{ for all } \xi \in S_N\}.$$

Here $a(\xi)$ is a continuous function on S_N which can be chosen as the reciprocal of half the width of E in the direction ξ. The function $a(\xi)$ is unique if E has a tangent plane at every boundary point. We now define

$$(1.2) \qquad F(\xi, z) = a(\xi)\xi \cdot z + \sqrt{(a(\xi)\xi \cdot z)^2 - 1}.$$

Here we always choose the sign of the root function to make $|F| \geq 1$. This choice makes, for a fixed ξ, $F(\xi, z)$ into a holomorphic mapping of $\{z : a(\xi)\xi \cdot z \notin [-1, 1]\}$ onto the complement of the unit disc in $\mathbf{C}$, with $|F(\xi, z)| \leq C(|z|+1)$.

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