ON SOME CLASSES OF FIRST-ORDER DIFFERENTIAL SUBORDINATIONS

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1. Introduction. Let f and F be analytic in the unit disk U. The function f is subordinate to F, written $f \prec F$ or $f(z) \prec F(z)$, if F is univalent, $f(0) = F(0)$ and $f(U) \subset F(U)$.

In two previous articles [1 and 5] the authors investigated properties of the *Briot–Bouquet differential subordination*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z). \tag{1}$$

This first-order differential subordination has many interesting applications in the theory of univalent functions [see 1, 5, and 8]. For special values of β, γ and h it has been investigated by many other authors.

The Briot–Bouquet differential subordination is but a special case of the general theory of differential subordinations introduced in [4, Section 4]. Restricting our attention to first-order differential subordinations, if $\psi: \mathbf{C}^2 \to \mathbf{C}$ is analytic in a domain D, if h is univalent in U, and if p is analytic in U with $(p(z), zp'(z)) \in D$ when $z \in U$, then p is said to satisfy the *first-order differential subordination*

$$\psi(p(z), zp'(z)) \prec h(z). \tag{2}$$

If we take $\psi(r,s) = r + s/(\beta r + \gamma)$ then the Briot–Bouquet differential subordination (1) can be written in the form (2).

Condition (2) is a generalization to complex function theory of the concept of first-order differential inequalities in real function theory. The similarity of the symbols $\prec$ and $<$ is appropriate since both represent an inclusion relation. In the real case there are many applications that require the finding of bounds on the function p satisfying the differential inequality (2) with $\prec$ replaced by $<$. This is also the case for differential subordinations. We now repeat the following definitions of dominant and best dominant from [4, Definition 4], here restricted to the first-order case.

DEFINITION 1. The univalent function q is said to be a *dominant* of the differential subordination (2) if $p \prec q$ for all p satisfying (2). If $\tilde{q}$ is a dominant of (2) and $\tilde{q} \prec q$ for all dominants q of (2), then $\tilde{q}$ is said to be the *best dominant* of (2). (Note that the best dominant is unique up to a rotation of U.)

In this article we determine dominants and best dominants of first-order differential subordinations. The special case of the Briot–Bouquet differential

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