APPROXIMATE FIBRATIONS ON TOPOLOGICAL MANIFOLDS

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1. Introduction. The main results of this paper deal with parameterized families of approximate fibrations. Approximate fibrations were introduced by Coram and Duvall in [4] as a generalization of both Hurewicz fibrations and cell-like maps. These maps have been studied by many authors, not only because of their intrinsic interest, but also because they arise naturally in certain problems concerning topological manifolds. Our main results are applied herein to study the relationship of approximate fibrations to bundles and fibrations, the local connectivity of spaces of bundles and fibrations, and the homotopy relation in the space of controlled homotopy topological structures on a fibration.

Let E and B be locally compact separable metric ANRs, let $p: E \to B$ be a proper map (i.e., inverse images of compacta are compact), let α be an open cover of B, and let C be a subset of B. We say that p is an *α-fibration over C* provided that given any X and maps $F: X \times [0,1] \to C$ and $f: X \to E$ for which $F(x,0) = pf(x)$, then there exists a map $\tilde{F}: X \times [0,1] \to E$ such that $\tilde{F}(x,0) = f(x)$ and $p\tilde{F}$ is α-close to F. If $C = B$, then p is called an *α-fibration.* And if p is an α-fibration for every open cover α of B, then p is an *approximate fibration.* If $\epsilon > 0$, then ϵ also denotes the open cover of B by balls of diameter ϵ. Thus, we also speak of *ϵ-fibrations.*

In this paper a *fiber preserving* (*f.p.*) map is a map which preserves the obvious fibers over an n-simplex Δ. Specifically, if $\rho: X \to \Delta$, $\sigma: Y \to \Delta$, and $f: X \to Y$ are maps, then f is f.p. if $\sigma f = \rho$. Usually the maps ρ and σ will be understood to be some natural projections and will not be explicitly mentioned. If $p: E \times \Delta \to B \times \Delta$ is a f.p. map, then f is an approximate fibration if and only if $f_t: E \to B$ is an approximate fibration for each t in Δ. This can be derived from [5].

A *manifold* will be understood to mean a topological manifold which possesses a handlebody decomposition. It is now known that this includes all topological manifolds except nonsmoothable 4-manifolds (see [19]).

Our first main result is a parameterized version of a theorem of Chapman [2, Theorem 1]. It enables one to detect which parameterized families of maps can be deformed to a close-by parameterized family of approximate fibrations.

DEFORMATION THEOREM. *Let B be a polyhedron, let $m \geq 5$, let Δ be an n-simplex, and let α be an open cover of B. There exists an open cover β of B so that if M is an m-manifold without boundary and $f: M \times \Delta \to B \times \Delta$ is a f.p. map such that $f_t: M \to B$ is a β-fibration for each t in Δ and an approximate fibration for each t in $\partial\Delta$, then there is a f.p. approximate fibration $\tilde{f}: M \times \Delta \to B \times \Delta$ such that $\tilde{f}_t$ is α-close to f_t for each t in Δ and $\tilde{f} \mid M \times \partial\Delta = f \mid M \times \partial\Delta$.*

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