QUASICONFORMAL ANALOGUES OF THEOREMS OF KOEBE AND HARDY-LITTLEWOOD

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Dedicated to Professor George Piranian on his seventieth birthday

1. Introduction. Suppose that D is a domain in euclidean n-space $\mathbf{R}^n$. We say that D is *uniform* if there exist positive constants a and b such that each pair of points $x_1, x_2 \in D$ can be joined by a rectifiable arc $\gamma \subset D$ for which

$$l(\gamma) \le a|x_1 - x_2| \tag{1.1}$$

and

$$\min_{j=1,2} l(\gamma_j) \le b d(x, \partial D) \tag{1.2}$$

for each $x \in \gamma$; here $l(\gamma)$ denotes the length of γ and γ_1, γ_2 the components of $\gamma \setminus \{x\}$.

Suppose next that D and D' are domains in $\mathbf{R}^n$ and that $f: D \to D'$ is K-quasiconformal with Jacobian J_f. Then $\log J_f$ is integrable over each ball $B \subset D$ and we set

$$(\log J_f)_B = \frac{1}{m(B)} \int_B \log J_f \, dm. \tag{1.4}$$

In particular, for each $x \in D$ we let

$$a_f(x) = \exp\left(\frac{1}{n}(\log J_f)_{B(x)}\right), \tag{1.5}$$

where $B(x) = B(x, d(x, \partial D))$, the open ball with center x and radius equal to the distance $d(x, \partial D)$ from x to ∂D. If $n = 2$ and f is conformal in D, then $\log J_f$ is harmonic,

$$(\log J_f)_{B(x)} = \log J_f(x) = 2 \log|f'(x)|$$

and hence $a_f(x) = |f'(x)|$.

We observed recently in [1] that for certain distortion properties of quasiconformal mappings the function a_f plays a role exactly analogous to that played by $|f'|$ when $n = 2$ and f is conformal. We investigate this analogy further by establishing in this paper quasiconformal versions of the following two well-known results due to Koebe [8, p. 22] and Hardy-Littlewood [6], respectively.

1.6. THEOREM. *Suppose that D and D' are domains in $\mathbf{R}^2$. If $f: D \to D'$ is conformal, then*

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