# ANALYTIC FUNCTIONS OF FINITE VALENCE, WITH APPLICATIONS TO TOEPLITZ OPERATORS

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*Dedicated to George Piranian on the occasion of his retirement*

**1. Introduction.** This paper concerns the extent to which the valence of a function analytic on the unit disc determines the form of that function. If $f$ is analytic on $\mathfrak{U} = \{|z| < 1\}$ and $w \in \mathbf{C}$, then the valence of $f$ at $w$, denoted $v_f(w)$, is the number of solutions $z \in \mathfrak{U}$ of $f(z) = w$, counting multiplicities. In [2], Baker, Deddens, and Ullman show that if $f$ is an entire function, and if $k$ is the smallest nonzero value of $v_f$, then $f(z) = h(z^k)$ for some entire function $h$. They ask whether the appropriate analogue of this result holds for functions which are not entire, with the role of $z^k$ played by a $k$-fold Blaschke product. We construct an example showing that the answer is no.

Our main effort concerns the study of pairs of functions whose valences are related. We prove a conjecture of Lee Rubel concerning entire functions complementing the Baker, Deddens, and Ullman result and show that it, too, fails if the functions are not entire. Given functions $f$ and $g$ with identical, finite valence functions, we investigate two structural relationships which may hold between them: One concerns the existence of a homeomorphism of the unit circle $\mathbf{T}$ which transforms the boundary values of $f$ to those of $g$. The other concerns the existence of a common function $h$ from which both $f$ and $g$ are obtained by composition. We show that the second relationship holds if and only if the first holds with a distinguished type of homeomorphism.

The paper concludes with applications of our results to the study of Toeplitz operators. A new condition is added to those of Carl Cowen [5] on similarity of analytic Toeplitz operators. This does not extend to rational Toeplitz operators; indeed, it leads to an example contradicting some published results concerning their similarity. A closer look at this example suggests the possible relevance of ideas used in the analytic case to a correct formulation. We also point out other implications of our work in the study of Toeplitz operators, including an answer to a question of Thomson [8] on commutants.

Throughout the paper we mention open questions which our work suggests. A word concerning terminology: The functions we study are defined on the unit disc $\mathfrak{U}$. To say that a function $f$ is entire, for instance, means that $f$ is the restriction to $\mathfrak{U}$ of an entire function.

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