ON CERTAIN TRANSCENDENTAL NUMBERS

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In this note we deduce some consequences of the Lindemann–Weierstrass theorem and the Gelfond–Schneider theorem. In a simple fashion we prove certain pairs of numbers are algebraically independent. We also show certain number classes contain only transcendental numbers. Examples include the algebraic independence of $e^2 \cos 3$ and $e^2 \sin 3$ and the transcendence of

$$\log \sin 2, \quad \sin \log 2, \quad \cos^{-1}(\exp(2)), \quad \exp(\cos^{-1} 2), \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\cos(\sqrt{2}n)}{n}.$$

One form of the Lindemann–Weierstrass theorem [1, p. 20] states that if $\alpha_1, \ldots, \alpha_n$ are algebraic, then $e^{\alpha_1}, \ldots, e^{\alpha_n}$ are algebraically independent if and only if $\alpha_1, \ldots, \alpha_n$ are linearly independent over the rationals. It follows at once that e^{α} is transcendental for all non-zero algebraic α. But what about the arithmetic nature of the real and imaginary parts of e^{α}? The transcendence of e^{α} insures that at least one of $\mathrm{Re}(e^{\alpha})$ and $\mathrm{Im}(e^{\alpha})$ is transcendental. But much more is true if the real and imaginary parts of α are both non-zero. In this case, a corollary of the following theorem shows that $\mathrm{Re}(e^{\alpha})$ and $\mathrm{Im}(e^{\alpha})$ are in fact algebraically independent.

THEOREM 1. *Suppose α and β are algebraic numbers. Then $e^{\alpha} \cos \beta$ and $e^{\alpha} \sin \beta$ are algebraically independent if and only if α and βi are linearly independent over the rationals.*

Proof. Let K denote the field of algebraic numbers. Then the following implications hold:

$e^{\alpha} \cos \beta$ and $e^{\alpha} \sin \beta$ are algebraically dependent.

$\Leftrightarrow$ Tr. $\deg._K K(e^{\alpha} \cos \beta, e^{\alpha} \sin \beta) \leq 1$.

$\Leftrightarrow$ Tr. $\deg._K K(e^{\alpha} \cos \beta + ie^{\alpha} \sin \beta, e^{\alpha} \cos \beta - ie^{\alpha} \sin \beta) \leq 1$.

$\Leftrightarrow$ Tr. $\deg._K K(e^{\alpha+\beta i}, e^{\alpha-\beta i}) \leq 1$.

$\Leftrightarrow$ $e^{\alpha+\beta i}$ and $e^{\alpha-\beta i}$ are algebraically dependent.

$\Leftrightarrow$ $\alpha + i\beta$ and $\alpha - i\beta$ are linearly dependent over the rationals (by the Lindemann–Weierstrass theorem).

$\Leftrightarrow$ α and βi are linearly dependent over the rationals.

The proof is complete. □

COROLLARY. *Suppose α is an algebraic number whose real and imaginary parts are both non-zero. Then the real and imaginary parts of e^{α} are algebraically independent.*

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