THE GAUSS–BONNET THEOREM FOR 2-DIMENSIONAL SPACETIMES

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The purpose of the present note is to extend the classical Gauss–Bonnet formula

$$\int_\Gamma k_g\, ds + \iint_D K\, dA + \sum_i \Theta_i = 2\pi$$

for a region D with boundary Γ on a 2-dimensional Riemannian manifold to the case of a 2-dimensional Lorentzian manifold. Such an extension becomes possible by refining the notion of angle in the Lorentzian plane which was defined in our previous paper [2].

Section 1 deals with the definition and properties of angle in a 2-dimensional spacetime and illustrates a special case of the formula dealing with the term $\sum_i \Theta_i$. In Section 2 we prepare needed facts for the terms $\int_\Gamma k_g\, ds$ and $\iint_D K\, dA$ and state the formula. The proof is given in Section 3 together with a concluding remark.

1. Angle in a spacetime. Following [6, pp. 24–27] we mean by a 2-dimensional spacetime a connected, 2-dimensional, oriented and time-oriented Lorentzian manifold (M, g). Thus M admits a globally defined unit timelike vector field which is future-pointing.

For each point x of M, the tangent space $T_x(M)$ is oriented and has a Lorentzian inner product together with time-orientation. For any unit timelike vector E in $T_x(M)$, we denote by $E^\perp$ the unique unit spacelike vector such that $g(E, E^\perp) = 0$ and such that the ordered basis $\{E, E^\perp\}$ is positively oriented. We say that a Lorentzian coordinate system $\{x_1, x_2\}$ in $T_x(M)$ is allowable if the vector $(0, 1)$ is a unit future-pointing timelike vector and $(0,1)^\perp = (1, 0)$.

Let X and Y be two unit timelike vectors which are future-pointing (or past-pointing). The angle from X to Y is defined to be the number u such that

$$\begin{bmatrix} \operatorname{ch} u & \operatorname{sh} u \\ \operatorname{sh} u & \operatorname{ch} u \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

where (x_1, x_2) and (y_1, y_2) are the components of X and Y, respectively, with respect to an allowable coordinate system. The number u is independent of the choice of an allowable coordinate system, as can be easily seen. We shall denote the angle from X to Y by (X, Y). The angle (Y, X) is equal to $-(X, Y)$. We have also $(-X, -Y) = (X, Y)$.

We now want to define the angle (X, Y) in the case where X is a future-

Received June 13, 1983. Revision received September 22, 1983.

The first author was supported by a fellowship of Consejo de Investigaciones Científicas y Técnicas de la República Argentina.

Michigan Math. J. 31 (1984).