DERIVATIONS FROM SUBALGEBRAS OF SEPARABLE C^*-ALGEBRAS

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1. Introduction. Let A be a C^*-algebra, $M(A)$ its multiplier algebra, B a C^*-subalgebra of A. Suppose $\delta: B \to A$ is a *derivation* of B into A, i.e., a linear map for which $\delta(ab) = a\delta(b) + \delta(a)b$, for all $a, b \in B$. In many important applications, one wishes to know if δ is *inner in* $M(A)$, i.e., if there is an element m of $M(A)$ for which $\delta(b) = mb - bm$, for all $b \in B$. Akemann and Johnson [1] have pointed out in particular the importance of investigating those pairs (B, A) as above for which every derivation of B into A is inner in this sense. A first step in such an investigation would consist of studying the C^*-algebras A for which *all* such pairs (B, A) have this property. We formalize this by saying that a C^*-algebra A is *hereditarily cohomologically trivial* (HCT for short) if for each C^*-subalgebra B of A and each derivation $\delta: B \to A$, there is a multiplier m of A for which $\delta(b) = mb - bm$, for all $b \in B$.

In [6], the authors determined the structure of the HCT C^*-algebras with continuous trace. The only other class of HCT algebras known to us are the finite von Neumann algebras, a result due to Erik Christensen [3, §5] (it is of course an outstanding open problem whether the algebra $B(H)$ of all bounded linear operators on a Hilbert space H is HCT). The HCT algebras are evidently contained in the class of C^*-algebras for which every derivation $\delta: A \to A$ is inner in $M(A)$, and Elliott [5] and Akemann and Pedersen [2] determined the structure of the separable C^*-algebras with this latter property. In the paper before the reader, we will determine the structure of the separable C^*-algebras which are HCT. It turns out that the separable HCT algebras form a rather restricted class; in fact the only simple, separable HCT algebras are the algebras of compact operators on a separable Hilbert space, usually referred to as the *elementary* C^*-algebras. More precisely, we will prove:

THEOREM 1.1. *Let A be a separable C^*-algebra. Then A is HCT if and only if A has a direct sum decomposition of the form $A_1 \oplus A_2$, where A_1 is a commutative algebra and A_2 is the restricted direct sum of a (possibly finite) sequence of separable elementary C^*-algebras.*

2. Proof of Theorem 1.1. We begin with a lemma which is no doubt well-known to the experts, but for which, in the interest of clarity and completeness, we provide a proof (it is stated without proof in the argument of Lemma 3.1 of [7]).

LEMMA 2.1. *Let A be a C^*-algebra, p a closed, central projection in the enveloping von Neumann algebra A^{**} of A. Then pA^{**} is naturally isomorphic to the enveloping von Neumann algebra $(pA)^{**}$ of pA.*

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