DISTANCE ESTIMATES AND PRODUCTS OF TOEPLITZ OPERATORS

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1. Introduction. In this paper we establish some results concerning distance estimates. One of these results produces an equivalent condition to the Axler-Chang-Sarason-Volberg theorem. In order to state our results more precisely, we fix some notations which will be used throughout the paper.

By D we will denote the open unit disc in the complex plane, and by ∂D its boundary. Let L^∞ denote the algebra of bounded measurable functions with respect to the Lebesgue measure on ∂D, and H^∞ denote the subalgebra of L^∞ consisting of all bounded analytic functions in D. We identify L^∞ with $C(X)$, the space of continuous functions on X, where X is the maximal ideal space of L^∞. The algebra $H^\infty + C$ is a closed subalgebra of L^∞; here $C = C(\partial D)$. It is known [14] that $H^\infty + C$ is the smallest closed subalgebra of L^∞ which contains H^∞. A closed subalgebra B of L^∞ which contains H^∞ is usually called a Douglas algebra. The maximal ideal space of B is denoted by $M(B)$. The reader is referred to [16] and [10] for the theory of Douglas algebras and to [9] for uniform algebras. The largest C^*-subalgebra of $H^\infty + C$ will be denoted by QC. Thus $QC = (H^\infty + C) \cap \overline{(H^\infty + C)}$, where bar denotes complex conjugation. The sets in the Shilov decomposition [19] of $M(L^\infty)$ associated with $H^\infty + C$ will be called QC-level sets. For $\phi \in M(H^\infty + C)$ the support of the representing measure for ϕ is called a support set. If A is a closed subspace of a Banach space Y and $x \in Y$, then $\operatorname{dist}(x, A) = \inf\{\|x - y\| : y \in A\}$. The annihilator of A in Y^* will be denoted by $A^\perp$, and $\operatorname{Ext}(A^\perp)$ denotes the set of the extreme points of $\operatorname{ball}(A^\perp)$. If B is a subset of Y, then $\operatorname{co}(B)$ denotes the convex hull of B.

The following results will be established.

THEOREM 1. *If A and B are two Douglas algebras such that $H^\infty + C = A \cap B$, then* $\operatorname{dist}(h, H^\infty + C) = \max\{\operatorname{dist}(h, A), \operatorname{dist}(h, B)\}$ *for all h in L^∞. Conversely, if the above condition is true then $H^\infty + C = A \cap B$.*

This result produces an equivalent condition to the Axler-Chang-Sarason-Volberg theorem. A proof of Theorem 1 appears in Section 2.

THEOREM 2. *Let A and B be two closed subalgebras of $C(X)$, where X is a compact Hausdorff space. Then the following conditions are equivalent:*

(1) $\operatorname{dist}(f, A \cap B) = \max\{\operatorname{dist}(f, A), \operatorname{dist}(f, B)\}$ *for all f in $C(X)$*;

(2) $\operatorname{Co}(\operatorname{ball} A^\perp \cup \operatorname{ball} B^\perp) = \operatorname{ball}(A \cap B)^\perp$;

(3) $\operatorname{Ext}((A \cap B)^\perp) \subset \operatorname{Ext}(A^\perp) \cup \operatorname{Ext}(B^\perp)$.

In Section 3, we show that condition (1) of Theorem 2 is not true in general.

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