C_0-FREDHOLM OPERATORS. III

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An operator T, acting on a Hilbert space, is said to be of class C_0 (cf. [6]) if T is a completely nonunitary contraction and $u(T)=0$ for some nonzero function u in H^∞. An operator T is said to have property (P) (cf. [1, part II]) if the equalities $\ker X=\{0\}$ and $\ker X^*=\{0\}$ are equivalent for every operator X in the commutant $\{T\}'$ of T. In the two preceding papers ([1]) we studied the multiplicative semigroup $\Phi(T',T)$ of C_0-Fredholm operators, associated with a given pair (T',T) of operators of class C_0. We recall that $\Phi(T',T)$ consists of those operators X intertwining T' and T $(T'X=XT)$ with the following properties:

(A) $T\mid\ker X$ and $T_{\ker X^*}(=(T^*\mid\ker X^*)^*)$ have property (P); and

(B) the mapping $X_*:\mathfrak{M}\mapsto(X\mathfrak{M})^-$, $\mathfrak{M}\in\operatorname{Lat}(T_{(\ker X)^\perp})$ is an isomorphism of $\operatorname{Lat}(T_{(\ker X)^\perp})$ onto $\operatorname{Lat}(T\mid(\operatorname{ran}X)^-)$.

Here, as usual, $(\mathfrak{M})^-$ stands for the closure of the set $\mathfrak{M}$. When $T'=T$, we use the notation $\Phi(T)$ for $\Phi(T',T)$ and note that $\Phi(T)$ is contained in $\{T\}'$. If T is the zero operator on $\mathcal{H}$, then $\Phi(T)$ coincides with the familiar class of Fredholm operators on $\mathcal{H}$.

In [1, part I, Lemma 3.3] we proved that the operators $T\mid\ker X$ and $T_{\ker X^*}$ are quasisimilar provided that T is of class C_0 and X belongs to the bicommutant $\{T\}''$ of T (cf. also [9]). We also know from [1, part II] that property (P) is a quasisimilarity invariant in the class C_0. Therefore, in order to verify that an operator X in $\{T\}''$ is C_0-Fredholm, it suffices to verify (B) and half of (A). In this paper we prove that condition (B) is a consequence of (A) for X in $\{T\}''$, thus establishing the following result.

THEOREM 1. *Let T be an operator of class C_0 and let $X\in\{T\}''$. Then X is C_0-Fredholm if and only if $T\mid\ker X$ has property* (P).

We had previously noted (cf. [1, part I, Proposition 3.5]) that Theorem 1 is true in case $\ker X=\{0\}$. Observe that property (B) is not satisfied for every X in $\{T\}''$ even in the case of nilpotent operators T; this follows from the discussion given below of C_0-Fredholm operators in the case when T is an algebraic operator (cf. Example 9 below).

Let T be an arbitrary operator of class C_0, and let m denote the minimal function of T. It follows from results of [3] and [2] that for every X in $\{T\}''$ there exist functions u,v in H^∞ such that v and m are relatively prime $(v\wedge m=1)$ and

$$v(T)X=u(T)$$

or, using the notation of [6], $X=(u/v)(T)$. It follows from the proof of the main theorem in [8] that v can be chosen independently of X.

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