DO ALMOST FLAT MANIFOLDS BOUND?

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0. Introduction. A closed connected Riemannian manifold M^n is said to be flat if its sectional curvatures are all zero. Since the (real) Pontrjagin classes are computable in terms of the sectional curvatures, a flat Riemannian manifold M has zero (real) Pontrjagin classes and hence zero Pontrjagin numbers when M is orientable. This observation was part of the reason for the conjecture that such a manifold is a boundary, i.e., that there exists a compact manifold W^{n+1} such that $\partial W = M$. (Recall that Thom had shown that a manifold bounds if and only if all its Stiefel–Whitney numbers vanish and Wall showed an orientable manifold is an oriented boundary if in addition all its Pontrjagin numbers vanish.) Recently, Hamrick and Royster, after partial results by Marc Gordon (cf. [6], [3]), have verified this conjecture.

In this paper, we conjecture that any almost flat manifold (whose definition is recalled below) bounds and give some partial results on this extended conjecture. We will rely heavily on the earlier methods of Gordon, Hamrick and Royster. Let $b(\ ,\ )$ be a Riemannian metric on a compact manifold M^n, let $d(M^n, b)$ denote the diameter of M^n with respect to $b(\ ,\ )$ and let $c(M^n, b)$ denote the maximum of the absolute values of the sectional curvatures of M^n relative to $b(\ ,\ )$. Following the terminology introduced by Gromov [5], an almost flat structure on a closed connected smooth manifold M^n is a sequence of Riemannian metrics $b_i(\ ,\ )$, where $i = 1, 2, \ldots$, such that

(0.1) (a) $\lim_{i \to \infty} c(M^n, b_i) = 0$ and

(b) $\{d(M^n, b_i) : i = 1, 2, \ldots\}$ has a finite upper bound.

CONJECTURE 1. *If M^n supports an almost flat structure, then there exists a compact smooth manifold W^{n+1} such that $\partial W^{n+1} = M^n$.*

This conjecture is geometrically motivated by work of Gromov [5]. He showed that if M^n supports an almost flat structure then M^n has a finite sheeted cover that is a nilmanifold and consequently the Pontrjagin classes of M^n vanish since nilmanifolds are parallelizable. Recall a nilmanifold is the quotient of a (connected) simply connected nilpotent Lie group by a discrete cocompact subgroup. In fact, a second result of Gromov [4] (which uses Margulis' lemma) suggests the above conjecture could possibly be strengthened as follows.

CONJECTURE 2. (a) *If M^n is a flat Riemannian manifold, then $M^n = \partial W^{n+1}$, where $W - \partial W$ supports a complete hyperbolic structure (constant negative sectional curvatures) with finite volume.*

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