THE FAILURE OF L^p ESTIMATES FOR HARMONIC MEASURE IN CHORD-ARC DOMAINS

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Let D be a bounded domain in the complex plane whose boundary ∂D is the image of a simple closed rectifiable curve. For $\zeta, \zeta' \in \partial D$, denote by $\sigma(\partial D; \zeta, \zeta')$ the length of the shorter arc of ∂D with endpoints ζ and ζ'. D is said to be a *chord-arc* domain if there is a constant C such that for every $\zeta, \zeta' \in \partial D$,

$$\sigma(\partial D; \zeta, \zeta') \leqslant C|\zeta - \zeta'|.$$

For each $p>0$, the Hardy class H^p is the collection of analytic functions F on the unit disc in the complex plane satisfying

$$\sup_{r<1} \mu_p(r,F) < \infty, \quad \text{where} \quad \mu_p(r,F) = \frac{1}{2\pi} \int_0^{2\pi} |F(re^{i\theta})|^p \, d\theta.$$

Our purpose is to comment on a theorem of Lavrentiev [8], namely

THEOREM 1. *For any constant C, there exists $p>0$ such that if f is a conformal mapping of the unit disc onto D and D is a chord-arc domain with constant C, then $1/f' \in H^p$.*

This result has received considerable attention recently ([1], [4], [6], [9]) by virtue of its link to real-variable lemmas of John-Nirenberg type and to the boundedness of the Cauchy integral on curves. In the closely related special case in which ∂D is given locally as the graph of a Lipschitz function, the corresponding conformal mapping f satisfies $1/f' \in H^1$ independent of the Lipschitz constant. The same is true of another simple example, a logarithmic spiral. For these reasons, Jerison and Kenig [6] and Baernstein [1] asked if Theorem 1 is valid with some exponent p independent of C, and in particular for $p=1$. We will show here that it is not.

THEOREM 2. *For any $p>0$, there exists a chord-arc domain D for which $1/f' \notin H^p$ for any conformal mapping f of the unit disc onto D.*

Jones and Zinsmeister have independently given another proof of this theorem [7].

Background and notation. We will reformulate Theorem 2 in terms of harmonic measure and state some standard results needed in the proof.

A well known consequence of Jensen's inequality is that

(1) *$\mu_p(r,F)$ is an increasing function of r for $r<1$* [10: p. 273].

If $F \in H^p$, then $F(e^{i\theta}) = \lim_{r \to 1} F(re^{i\theta})$ exists for almost every $\theta \in [0, 2\pi)$ and

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