SMOOTH FREE INVOLUTIONS ON HOMOTOPY $4k$-SPHERES

Ronald Fintushel and Ronald J. Stern

1. Introduction. In [7] we constructed a homotopy real projective 4-space Q^4 not s-cobordant to the standard real projective 4-space $\mathbf{R}P^4$ with the property that the 2-fold cover of Q was diffeomorphic to the 4-sphere S^4, thereby constructing a smooth exotic involution on S^4. In order to determine that Q was not s-cobordant to $\mathbf{R}P^4$ we introduced an invariant which was a $\mathbf{Q}/\mathbf{Z}$-linear combination of the μ-invariant of an almost-framed characteristic homology $\mathbf{R}P^3$ and the α-invariant of its 2-fold cover. In this paper we generalize this invariant to study smooth manifolds of the homotopy type of $\mathbf{R}P^{4k}$, $k>1$, or equivalently to study smooth free involutions on homotopy $4k$-spheres. This invariant, defined in §2 and called ρ, is again essentially a $\mathbf{Q}/\mathbf{Z}$-linear combination of the Eells-Kuiper μ-invariant of a spin structure on a characteristic homotopy projective space and the Atiyah-Singer α-invariant of its double cover.

In §3 we show that (except for a single exceptional case $\rho=1/4$), every normal cobordism class of homotopy $\mathbf{R}P^{4k-1}$'s which contains a representative whose double cover is the standard sphere S^{4k-1} gives rise to at least two homotopy $\mathbf{R}P^{4k}$'s which are distinguished by ρ. In §4 we construct normal cobordism classes of homotopy $\mathbf{R}P^{4k-1}$'s which contain representatives whose double cover is S^{4k-1}, and we distinguish these normal cobordism classes by a difference invariant, essentially 2ρ, introduced in §2. This then yields twice as many homotopy $\mathbf{R}P^{4k}$'s.

In Appendix I we give an explicit calculation of the μ-invariants of $\mathbf{R}P^{4k-1}$ which is used in §3 and §4.

We obtain no more smooth homotopy $\mathbf{R}P^{4k}$'s than claimed earlier by, for instance, Giffen [8] or Löffler [12]. The strength of our approach lies in the simplicity of the invariant ρ and its utility in distinguishing specific examples. This is exemplified by the explicit construction of all the homotopy $\mathbf{R}P^8$'s and homotopy $\mathbf{R}P^{12}$'s in Appendix II.

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2. The invariants. Given a closed manifold Q^{4k-1} of the homotopy type of $\mathbf{R}P^{4k-1}$ whose double cover $\tilde{Q}$ is diffeomorphic to S^{4k-1} we construct a homotopy $\mathbf{R}P^{4k}$ as follows. Identify $\tilde{Q}$ with the boundary of the $4k$-ball B^{4k} and let

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