ON MEROMORPHIC FUNCTIONS WHOSE IMAGES CONTAIN A GIVEN DISC

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1. Introduction. Let S denote the well-known family of functions, f, that are analytic and univalent in $D=\{z: |z|<1\}$ and have the normalization $f(0)=0$, $f'(0)=1$. Netanyahu [6], [7] introduced the classes $S(d)=\{f\in S: d=d_f\}$ where d_f is defined by

$$d_f=\inf\{|\alpha| : f(z)\neq\alpha\}. \tag{1.1}$$

He studied the relation between the second coefficient and d. This concept has been extended in several directions in [3], [8].

We define M to be the set of all normalized univalent meromorphic functions $f(z)=z+a_2z^2+\cdots$ in D. For each function f in M, we define d_f by (1.1), and let $M(d)=\{f\in M: d\leqslant d_f\}$.

For each $p\in(0,1)$, Goodman [2] introduced the classes

$$U(p)=\{f\in M: f(p)=\infty\}.$$

He conjectured that $k_p(z)=pz/[(p-z)(1-pz)]$ is the extremal function for the coefficient problem in $U(p)$. Ever since, several papers have been written on this conjecture, each obtaining partial results. Among these, we mention the ones by Jenkins [4], and Kirwan and Schober [5].

It follows from the minimum modulus problem of Fenchel [1], that $U(p)$ is a subset of $M(d)$ when $d=p/(1+p)^2$, and for all d we have $S(d)\subseteq M(d)$. Therefore, it seems natural to study the classes $M(d)$.

In this paper, we solve the coefficient problem for $M(d)$, and moreover we are able to find the extreme points of the closed convex hull of $M(d)$. The coefficient bounds we obtain for $M(d)$ are valid but, of course, are not sharp for the subclasses $U(p)$ and $S(d)$.

2. Extreme points of the closed convex hull of $M(d)$. We begin with some preliminary results.

LEMMA 1. *Suppose $f\in M(d)$, then $d|z|\leqslant|f(z)|$ for all $|z|<1$.*

Proof. Let us define $\phi(\eta)=f^{-1}(d\eta)$. Then $\phi(0)=0$ and $|\phi(\eta)|\leqslant 1$ for $|\eta|<1$. So, by the Schwarz Lemma, we obtain $|\phi(\eta)|\leqslant|\eta|$ for all $|\eta|<1$. However, this is equivalent to the conclusion of the lemma when $|f(z)|<d$. To see this, let $d\eta=f(z)$. Note that when $|f(z)|\geqslant d$, the conclusion holds trivially. □

LEMMA 2. *If $f\in M(d)$ has a pole at the point a, then $|a|\geqslant d$.*

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