HOMOTOPY EQUIVALENCES OF PUNCTURED MANIFOLDS

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Let M be a closed (smooth or PL) manifold of dimension $n \geqslant 2$, and let W be the punctured manifold obtained by removing from M the interior of a (smoothly or PL)-imbedded n-disc D, centered at the basepoint m of M. Not surprisingly, the group $G_1(M)$ of homotopy classes of basepoint-preserving degree 1 self-homotopy-equivalences of M is closely related to the group $G(W, \partial W)$ of homotopy classes (rel ∂W) of self-homotopy-equivalences of W that fix ∂W. The main theorem of this paper makes this precise. We begin by stating the theorem and outlining various applications.

The map $r: W \to W$ is a rotation about the boundary sphere of W, described in Section 1(b). Our main theorem is

THEOREM 3.2. *There is a central extension*

$$1 \longrightarrow K \longrightarrow G(W, \partial W) \longrightarrow G_1(M) \longrightarrow 1$$

where K is the subgroup of $G(W, \partial W)$ generated by $\langle r \rangle$. For $n \geqslant 3$, $K = 0$ or $\mathbf{Z}/2$ according as $\langle r \rangle = \langle 1_W \rangle$ or $\langle r \rangle \neq \langle 1_W \rangle$. For $n = 2$, $K = 0$ if M is the 2-sphere or real projective plane, otherwise $K \cong \mathbf{Z}$.

After proving this theorem in Section 3, we apply it to the problem of deforming homotopy equivalences to homeomorphisms, showing that every element of $G_1(M)$ contains PL homeomorphisms if and only if every element of $G(W, \partial W)$ does. In Section 4, we consider the case of M aspherical. In this case, $\langle r \rangle \neq \langle 1_W \rangle$, so $G(W, \partial W)$ is determined, at least up to extension, by $\pi_1(M, m)$. In Section 5, we apply Theorem 3.2 to describe the stabilizers of certain elements in finitely-generated free groups. In the final section, we show that when $M = T^n$, the n-dimensional torus, the exact sequence of Theorem 3.2 is isomorphic to a well-known sequence involving the Steinberg group $\mathrm{St}(n, \mathbf{Z})$. Thus, the extension need not be trivial.

In the first section, we will discuss a few preliminaries, including the fact that $G(W, \partial W)$ is a group. The main lemma, from which Theorem 3.2 follows easily, is proved in Section 2. The proof uses geometric constructions to simplify a homotopy between a self-homotopy-equivalence of M and the identity map of M.

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1. Preliminaries.

1.a. *Mapping spaces of manifolds.*

We will always work only with basepoint-preserving maps of M, and use the C-O topology on mapping spaces. If $A(N)$ is a space of mappings from a manifold N to itself, and $X \subset N$, let $A(N, X) = \{f \in A(N) : f|_X \text{ is the identity map } 1_X\}$. When the

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