p-SUBGROUPS OF COMPACT LIE GROUPS AND TORSION OF INFINITE HEIGHT IN $H^*(BG)$, II

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1. The main purpose of this paper is to describe the size of the p-torsion subring of $H^*(BG, \mathbf{Z})$ where G is a compact Lie group. We construct a subring of $H^*(BG, \mathbf{Z})$ which is a direct sum of reduced polynomial algebras over $\mathbf{Z}/p^i\mathbf{Z}$ for various i (Corollary 1.7). This is accomplished by using transfer results largely developed in Part 1 of this paper [2]. A detection result of Quillen implies this is the best possible result in a suitable sense (Corollary 1.8).

The integral cohomology of the classifying space of a compact Lie group is extremely complicated in general. Even the torsion free quotient ring may behave oddly. For example if $G = \mathrm{Spin}(12)$, the quotient ring is not isomorphic to the invariants of the cohomology of the classifying space of a maximal torus under the action of the Weyl group [4]. In theory if one knows the rational and $\bmod p$ cohomologies of BG, in addition to all the Bocksteins, one can derive a great deal of information about the integral cohomology of BG. This is often very difficult in practice however. One might also think that knowing $H^*(G)$ one could easily determine $H^*(BG)$ by a spectral sequence argument. The point is that the calculations become extremely difficult. The transfer techniques used here, however, are straightforward and produce a subring of interest.

A result we use several times is the following theorem of Quillen [5], which we shall summarize briefly. First there is a detection part of the theorem. This says that any non-nilpotent element in $H^*(BG, \mathbf{Z}/p\mathbf{Z})$ is detected on some maximal elementary abelian p-subgroup of G. The second result concerns the existence of non-nilpotent elements. Let $x \in \bigoplus H^*(BL, \mathbf{Z})$, where the direct sum is over all conjugacy classes of maximal elementary abelian p-subgroups of G, with L being a representative subgroup. If the coordinates of x are compatible with the obvious necessary conditions imposed by the relationship between the elementary abelian p subgroups of G and G, then some pth power of x is the image of an element of $H^*(BG, \mathbf{Z}/p\mathbf{Z})$.

We shall only use the detection part of Quillen's theorem in this paper. We note that Quillen and Venkov have given a short proof of this part of the theorem for finite G [6]. One may wonder to what extent the full theorem of Quillen can be used to prove results like Corollary 1.7 concerning the existence of p-torsion elements in $H^*(BG, \mathbf{Z})$ even though Quillen's theorem provides no information about the Bocksteins. The answer is somewhat technical. If a maximal elementary abelian p-subgroup has p-rank larger than the rank of G, Quillen's theorem can in fact be used to show the existence of p-torsion elements which are detected on this subgroup. This takes a little work however. If the p-rank of the maximal elementary abelian p-subgroup L is not greater than the rank of G, but L is not contained in a maximal

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