QUASIAFFINE TRANSFORMS OF OPERATORS

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1. Introduction. Let $\mathcal{H}$ and $\mathcal{H}'$ be separable, infinite dimensional, complex Hilbert spaces, and let $\mathcal{L}(\mathcal{H}',\mathcal{H})$ denote the space of all bounded, linear operators from $\mathcal{H}'$ to $\mathcal{H}$. If $\mathcal{H}'=\mathcal{H}$, we write $\mathcal{L}(\mathcal{H})$ in place of $\mathcal{L}(\mathcal{H}',\mathcal{H})$. An X in $\mathcal{L}(\mathcal{H}',\mathcal{H})$ is called a *quasiaffinity* if it has trivial kernel and dense range. An operator T' in $\mathcal{L}(\mathcal{H}')$ is said to be a *quasiaffine transform* of an operator T in $\mathcal{L}(\mathcal{H})$ (notation: $T' \prec T$) if there exists a quasiaffinity X in $\mathcal{L}(\mathcal{H}',\mathcal{H})$ such that $XT'=TX$. If both $T' \prec T$ and $T \prec T'$, then we say that T' and T are *quasisimilar*, and we write $T' \sim T$. These relations have already played a considerable role in operator theory (see, for example, [13]), and under certain additional hypotheses, one knows that $T' \prec T$ implies $T' \sim T$. This is true when T and T' are normal [13], and also when T and T' belong to the class C_0 in the terminology of [13], cf. [17]. Moreover, if $T' \prec T$ and T is *algebraic*, i.e., satisfies some polynomial equation, then T' is also algebraic and $T' \sim T$ [2], [19]. However, it has been known for a long time (cf. [12] and [16]) that the relation "$\prec$" is not reflexive, and that, in general, one may have $T' \prec T$ without T' inheriting many of the properties of T. For nonalgebraic operators this phenomenon is quite striking. Indeed, it was shown in [16] that if T is a nonalgebraic strict contraction having a cyclic vector, then the unilateral unweighted shift $S^{(\infty)}$ of infinite multiplicity is a quasiaffine transform of T. We will show in Section 3 that this result remains true if the hypothesis of possessing a cyclic vector is omitted. It will follow that every nonalgebraic operator in $\mathcal{L}(\mathcal{H})$ has a quasiaffine transform of the form $\alpha S^{(\infty)}$ for some positive number α.

In Section 4 we will use the main theorem of Section 3 and a result of Berg [4] to prove that every T in $\mathcal{L}(\mathcal{H})$ has a quasiaffine transform of the form $N+K$ where N is normal and K is a compact operator of arbitrarily small norm. Of course, as noted above, if T is algebraic, then T and $N+K$ are quasisimilar. Unfortunately the relation $N+K \prec T$ does not always imply quasisimilarity. Indeed, we show that a quasinilpotent operator which is quasisimilar to an operator of the form $N+K$ (with N normal and K compact) necessarily commutes with a nonzero compact operator, and one knows from [7] that there are quasinilpotent operators that do not commute with any nonzero compact operator.

Finally, we improve a result from [7] by showing that any operator T in $\mathcal{L}(\mathcal{H})$ whose essential spectrum $\sigma_e(T)$ is the singleton $\{0\}$ has a compact quasiaffine transform K. Again in this case the relation $K \prec T$ cannot be replaced by quasisimilarity.

We remark that the results in this paper were obtained in 1976 and 1977. Earlier versions of the paper appeared as INCREST preprints in those years, and the main results were announced in the Notices of the A.M.S. in 1978 (78T-B110).

2. Preliminaries. We will denote by $\mathbf{K}=\mathbf{K}(\mathcal{H})$ the ideal of compact operators in $\mathcal{L}(\mathcal{H})$ and by $\pi:\mathcal{L}(\mathcal{H})\to\mathcal{L}(\mathcal{H})/\mathbf{K}$ the canonical projection onto the Calkin

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