ON PARTIALLY TEICHMÜLLER BELTRAMI DIFFERENTIALS

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Introduction. If E is a measurable subset of a Riemann surface R, we consider two problems associated with quasiconformal mappings F from R to another Riemann surface R_1 with the property that F is conformal on $R-E$. The first problem is to characterize extremal mappings in the same Teichmüller class as F subject to the condition that they be conformal on $R-E$. Using a modified version of the "main inequality" of Reich and Strebel we can show that mappings F extremal in this sense have partially Teichmüller form. By this we mean there is an integrable holomorphic quadratic differential ϕ on the surface R_1 such that F^{-1} has Beltrami coefficient $k|\phi|/\phi$ on $F(E)$ and zero on $R_1-F(E)$. It also follows that such differentials are extremal. To a certain extent these results can be viewed as analogous to those presented by Reich in [6]. In Reich's notation we are treating the case where $b_1=0$, which Reich says has special interest and is not included in his own treatment.

The second problem we consider concerns mappings F which are Teichmüller trivial on R and which are also conformal on $R-E$. The space of Beltrami differentials of such mappings is denoted by $M_0(R,E)$. We are unable to show that $M_0(R,E)$ is connected but can show that given μ in $M_0(R,E)$ corresponding to F there is a mapping μ_1 in $M_0(R,E)$ corresponding to F_1 such that if μ_2 is the Beltrami coefficient for $F\circ F_1^{-1}$ then $\|\mu_i\|_\infty < \|\mu\|_\infty$ for $i=1$ and $i=2$. The result also gives real analytic parametric path of Beltrami differentials in $M_0(R,E)$ connecting μ to μ_1.

1. Preliminaries. Let R be a Riemann surface, C be the union of the ideal boundary curves of R and let σ be a closed subset of C. It is possible for either C or σ to be empty. When C is nonempty we consider it to be part of R so in this case R is a bordered Riemann surface.

Let $A(R,\sigma)$ be the space of integrable, holomorphic, quadratic differentials on R which are real with respect to real boundary parameters at points of $C-\sigma$. If R is of finite type with genus g, n boundary contours and k interior punctures and σ is a finite subset of C, then from the Riemann-Roch theorem one can show that the real dimension of $A(R,\sigma)$ is

$$6(g-1)+3n+2k+\operatorname{card}(\sigma)+\rho \tag{1}$$

where ρ is the real dimension of the continuous group of holomorphic homeomorphisms of R. ρ can be positive only in special cases when $g=1$ or 0.

Let $L(R)$ be the set of all Beltrami differentials on R. An element μ of $L(R)$ is an assignment of a measurable function μ^z to each local parameter z such that

(a) $\mu^z(z)\dfrac{d\bar z}{dz}=\mu^\zeta(\zeta)\dfrac{\overline{d\zeta}}{d\zeta}$ for any two parameters z and ζ with overlapping domains and

(b) $\|\mu\|_\infty=\sup\{\|\mu^z(z)\|_\infty \text{ for all } z\}<\infty$.

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