POLYNOMIAL VALUES AND ALMOST POWERS

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1. Introduction. For $n \in \mathbf{Z}$ and $m \in \mathbf{N}$ with $m \geqslant 2$ the m-free part of n is the smallest positive integer a with the property that $\pm n = ay^m$ for some $y \in \mathbf{Z}$. Let $F \in \mathbf{Z}[X]$ have at least two distinct zeros. We prove that if $a > 1$ is the m-free part of $F(x)$ for some $x \in \mathbf{Z}$ and some $m \in \mathbf{N}$ with $m \geqslant 2$ then a has a certain multiplicative structure (see Corollary 1), e.g. the greatest prime divisor $P(a)$ of a exceeds $c_1 \log\log a$, where c_1 is a positive number depending only on F. This includes the well known fact that $P(F(x)) > c_1 \log\log |F(x)|$ for $|F(x)| > 1$. Let $F \in \mathbf{Z}[X]$ have at least three simple zeros and suppose that $F(x) = \pm ab$, where $x \in \mathbf{Z}$, $a \in \mathbf{N}$ and b is some power, i.e. $b \in \{y^m \mid y \in \mathbf{Z}, m \in \mathbf{N}, m \geqslant 2\}$. We prove that a cannot be small in comparison to $|F(x)| : a > \frac{1}{2}\exp(c_1(\log\log(|F(x)|+3))^{1/5-\epsilon})$, where c_1 denotes a positive constant depending only on F and $\epsilon > 0$. This includes the well-known fact that there exist only finitely many $x \in \mathbf{Z}$ such that $F(x)$ is a power. We also show that these numbers a are not a product of primes which are very small with respect to $|F(x)| : P(a) > c_3 \log\log\log(|F(x)|+3)$, where $c_3 > 0$ depends only on F.

2. Let $F \in \mathbf{Z}[X]$ have at least two distinct zeros and let $a \in \mathbf{Z}$, $a \neq 0$.

It is well known (see, e.g., [1]) that there exist positive numbers ϵ_F and $c_F(a)$ such that $P(F(x)) > \epsilon_F \log\log|F(x)|$ for $|F(x)| > 1$ and such that if $F(x) = ay^m$, for certain $x, y \in \mathbf{Z}$ with $|y| > 1$ and some $m \in \mathbf{N}$, then $m \leqslant c_F(a)$. In the existing proofs for the second result, the first result is used. This is unnecessary, in fact the first result can be proved in the same manner as the second one, as can be seen in the proof of Theorem 1 in this section. We also give an upper bound $c_F(a)$ for m which is explicit in a. For completeness we state the results that we use in the proof of Theorem 1. These results can be found in [1] as Theorem A (= Proposition), Lemma C (= Lemma 1), while Lemma 2 is implicit in the proof of Theorem 1 in [1] (see also [2], Lemma 4.3).

PROPOSITION. *Let $\alpha_1, \ldots, \alpha_N$, where $N \geqslant 2$, be nonzero algebraic numbers. Let K be the smallest normal field containing $\alpha_1, \ldots, \alpha_N$ and put $d = [K:\mathbf{Q}]$. Let $A_1, \ldots, A_N$ $(\geqslant 3)$ be upper bounds for the heights of $\alpha_1, \ldots, \alpha_N$, respectively. Put $\Omega' = \prod_{j=1}^{N-1} \log A_j$, $\Omega = \Omega' \log A_N$. There exist positive numbers C_1 and C_2 such that for every $B \geqslant 2$ the inequalities*

$$0 < |\alpha_1^{b_1} \ldots \alpha_N^{b_N} - 1| < \exp(-(C_1 N d)^{C_2 N} \Omega \log \Omega' \log B)$$

have no solution in rational integers $b_1, \ldots, b_N$ with absolute values at most B.

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