REMARKS ON SUBSPACES OF H_p WHEN $0<p<1$

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1. Introduction. Let $\mathbf{T}$ be the unit circle in the complex plane and let Δ be the open unit disc. As usual H_p, $0<p<1$ denotes the quasi-Banach space of all functions $f:\Delta\to C$ analytic in Δ such that

$$\|f\|_p^p = \sup_{0<r\leqslant 1} \int_{\mathbf{T}} |f(rw)|^p\, dm(w) < \infty$$

where m is normalized Lebesgue measure on the circle. By considering boundary values H_p can be identified with a closed subspace of $L_p(\mathbf{T})$.

In this paper we give a number of results on the closed subspaces of H_p. Our first result is to show that H_p can have no complemented locally convex subspaces; this answers a question of Shapiro (see [7]). Indeed, we show that H_p cannot have any locally convex subspaces with the Hahn-Banach Extension Property (HBEP). A closed subspace M of a quasi-Banach space X has HBEP if every continuous linear functional on M can be extended to a continuous linear functional on X.

Next we consider special subspaces of the type $H_p(M)$ where M is a set of non-negative integers. Then $H_p(M)$ is the closed linear span of $\{z^m : m\in M\}$. We show that $H_p(M)$ can only have HBEP if it is thick in the sense that if

$$M=\{m_n : n=1,2\dots\} \quad \text{where} \quad m_1<m_2<m_3\dots$$

then $m_n\leqslant cn$ for some constant c. This again answers a question raised by Shapiro; Duren, Romberg and Shields [3] observed that $H_p(M)$ fails to have HBEP when M is a Hadamard gap sequence.

We also show that $H_p(M)$ is the range of a translation-invariant projection if and only if M is a finite union of arithmetic progressions modulo a finite set.

In the last section we discuss the nature of Banach subspaces of H_p. We conjecture that every Banach subspace of H_p has the Radon-Nikodym Property and show this is true for translation-invariant subspaces.

2. Preliminaries. We recall that a complex quasi-normed linear space X is called a quasi-Banach space and that if for some p, $0<p\leqslant 1$, the quasi-norm obeys the law

$$\|x_1+x_2\|^p \leqslant \|x_1\|^p + \|x_2\|^p \quad x_1,x_2\in X$$

then X is called a p-Banach space. The dual space of X will be denoted by X^*. If X^* separates the points of X, then the Mackey topology on X is the finest locally convex topology on X with the same dual space. This topology is a norm topology generated by $\operatorname{co}(U)$ where $U=\{x:\|x\|\leqslant 1\}$ is the unit ball of X. Let $|||\cdot|||$ be the associated

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