LOWER BOUNDS FOR THE EIGENVALUES OF RIEMANNIAN MANIFOLDS

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1. Introduction. Let M be a compact Riemannian manifold with boundary ∂M. Denote Δ_0 to be the Laplacian of M for functions.

Under the assumption that M was negatively curved [6], we gave lower bounds for the eigenvalues of Δ_0, subject to Dirichlet boundary conditions when $\partial M \neq \phi$. A main device employed was to obtain an upper bound for the trace of the heat kernel of M.

The present paper extends the work of [6] in several directions. First of all, we improve and simplify the elementary lemmas showing that an upper bound on the heat kernel gives lower bounds for the eigenvalues. The restriction of negative curvature is removed, giving lower bounds for the eigenvalues of Δ_0 on arbitrary M, assuming Dirichlet boundary conditions if $\partial M \neq \phi$.

In Section 4, we consider Laplacians Δ acting on Riemannian vector bundles $V \to M$. Upper bounds are obtained for the associated heat kernels. This implies upper bounds for the number of non-positive eigenvalues and also lower bounds for the positive eigenvalues of Δ. Interesting special cases include the Laplacian on Differential Forms and the Second Variation Operator of Minimal Submanifold Theory. In particular, if M is minimally imbedded, upper bounds are given for the nullity and index of M. One also obtains upper bounds for the betti numbers of M.

2. Heat kernels and eigenvalues. Let M be a compact Riemannian manifold with boundary ∂M. A Riemannian vector bundle $V \to M$ is a smooth vector bundle with metric and connection ∇ preserving that metric. The Bochner Laplacian of V is an invariantly defined second order differential operator $D : \Gamma(V) \to \Gamma(V)$, obtained by $D = \mathrm{Tr}(\nabla \circ \nabla)$. Here $\Gamma(V)$ denotes the smooth sections of V. More explicitly, D is given by the composition:

$$\Gamma(V) \xrightarrow{\nabla} \Gamma(V \otimes T^*M) \xrightarrow{\nabla} \Gamma(V \otimes T^*M \otimes T^*M) \to \Gamma(V)$$

where the last map is a contraction.

Suppose that E is a selfadjoint endomorphism of V and set $\Delta = -D + E$. If ∂M is empty, then Δ defines a unique selfadjoint operator in $L^2 V$. Otherwise, one must impose suitable boundary conditions. It is most typical to use either Dirichlet, $w(p) = 0$ for $p \in \partial M$, or Neumann $\nabla_\nu w(p) = 0$ for $p \in \partial M$, boundary conditions. Here $w \in \Gamma(V)$ and ν denotes a unit normal vector field along ∂M.

Since M is compact, the operator Δ has a finite number of non-positive eigenvalues $\mu_1 \leqslant \mu_2 \leqslant \ldots \mu_l \leqslant 0$ and an infinite collection of positive eigenvalues $0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots.$ The λ_i satisfy the asymptotic estimate of Minakshisundaram-Pleijel [11]:

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