A DUALITY THEOREM FOR HARMONIC FUNCTIONS

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Let D be a bounded open subset of $\mathbf{R}^n$ with C^∞ boundary, and let $h^\infty(\bar{D})$ denote the space of complex valued harmonic functions on D which are in $C^\infty(\bar{D})$. In this paper, we prove that the dual of the Frechet space $h^\infty(\bar{D})$ is the space $h^{-\infty}(D)$ of harmonic functions on D which satisfy finite growth conditions at the boundary. More precisely, a harmonic function g is in $h^{-\infty}(D)$ if and only if there are positive constants m and C such that $\text{Sup}\{|g(z)|\,d(z)^m : z \in D\} < C$ where $d(z)$ is the distance of z to bD, the boundary of D. In fact, we prove that $h^\infty(\bar{D})$ and $h^{-\infty}(D)$ are mutually dual via an extension of the usual $L^2(D)$ pairing.

This duality in conjunction with some classical results from potential theory allows us to prove an interesting theorem about the Poisson kernel $P(x,\theta)$ of the domain D. It is a classical fact that the operator $\phi \mapsto \int_{bD} P(x,\theta)\phi(\theta)\,d\sigma_\theta$ maps $C^\infty(bD)$ isomorphically onto $h^\infty(\bar{D})$. In this paper, we prove that the operator

$$h \mapsto \int_D h(x)P(x,\theta)\,dV_x,$$

when defined correctly, is an isomorphism between $h^{-\infty}(D)$ and $\mathfrak{D}'(bD)$.

A key step toward proving these results is the establishment of

LEMMA 1. *Suppose D is a smooth bounded domain in $\mathbf{R}^n$ and s is a positive integer. There is a positive integer $m = m(s)$ and a constant $C = C(s)$ such that if f and g are harmonic functions in $L^2(D)$, then*

$$\left|\int_D fg\right| \le C\Big(\sup_{z\in D,\ |\alpha|\le m} |\partial^\alpha f(z)|\Big)\Big(\sup_{z\in D} |g(z)|\,d(z)^s\Big).$$

Here, the symbol ∂^α is defined when $\alpha = (a_1, a_2, \ldots, a_n)$ is a multi-index as the differential operator

$$\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{a_1}\partial x_2^{a_2}\cdots\partial x_n^{a_n}}.$$

The constants m and C do not depend on f or g.

This lemma leads to the remarkable conclusion that if $f \in h^\infty(\bar{D})$ and $g \in h^{-\infty}(D)$, then $\int_D fg$ is a well defined quantity, even though $|fg|$ may be far from integrable.

Before we can state and prove our main theorem, we must establish some definitions and recall some facts from potential theory.

Throughout this paper, D will be a smooth bounded domain contained in $\mathbf{R}^n$. If s is a positive integer, we let $W^s(D)$ denote the usual Sobolev space of complex valued functions on D with norm $\|\ \|_s$ induced by the inner product

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