ORIENTATION-REVERSING PL INVOLUTIONS ON ORIENTABLE TORUS BUNDLES OVER S^1

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0. Introduction. In this paper we characterize the orientation-reversing PL involutions on orientable torus bundles over S^1. Let $M(b)$ denote the Seifert manifold of type $\{b; (n_2, 2)\}$ (see P. Orlik [12] or P. Conner and F. Raymond [3]). The space $M(0)$ is the only Seifert manifold of this type which admits an orientation-reversing PL involution (see [11] or Theorem B below). We also obtain a complete classification of these involutions on $M(0)$. Crucial to our study is the following, as well as of interest in its own right.

THEOREM A. (1) *If an orientable torus bundle over S^1 admits a* PL *embedding of a Klein bottle K, then it is homeomorphic to a Seifert manifold $M(b)$ for some b.*

(2) *A union of two twisted I-bundles over K (as an adjunction space) with infinite first homology group is homeomorphic to $M(b)$ for some b.*

We remark that each Seifert manifold $M(b)$ contains a Klein bottle. Thus Theorem A shows that the family of orientable torus bundles over S^1 which contain Klein bottles is identified with the Seifert manifolds $M(b)$. Recall that $M(b)$ and $M(b')$ are homeomorphic if and only if $b' = \pm b$.

In Theorems B, C we characterize the orientation-reversing PL involutions on orientable torus bundles over S^1. In order to do this, we need to define some fibered 3-manifolds. Let $\mathbf{R}$ be the set of real numbers and S^1 the set of complex numbers with norm 1. The 2-dimensional torus T^2 may be represented as $T^2 = \{(z_1, z_2) \mid z_1, z_2 \in S^1\}$. Let φ be a homeomorphism of T^2. We let $T^2 \times \mathbf{R}/\varphi$ denote the torus fiber bundle over S^1 obtained from $T^2 \times \mathbf{R}$ by identifying (x, t) with $(\varphi(x), t+1)$ for each $(x, t) \in T^2 \times \mathbf{R}$. The elements of $T^2 \times \mathbf{R}/\varphi$ are denoted by $[x, t]$. Define a homeomorphism $\bar{\varphi}: T^2 \longrightarrow T^2$ by $\bar{\varphi}(z_1, z_2) = (z_1^{\lambda p+1} z_2^{\lambda q}, z_1^{\lambda r} z_2^{\lambda s+1})$ where p, q, r, s are integers with $\left|\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right| = -1$ and $\lambda = p+s$. Observe that $\left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)^2 = \left(\begin{smallmatrix} \lambda p+1 & \lambda q \\ \lambda r & \lambda s+1 \end{smallmatrix}\right)$. We denote the space $T^2 \times \mathbf{R}/\bar{\varphi}$ by $\bar{M}\left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)$. Then two spaces $\bar{M}\left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)$ and $\bar{M}\left(\begin{smallmatrix} p' & q' \\ r' & s' \end{smallmatrix}\right)$, different from $T^2 \times S^1$, are homeomorphic if and only if the two matrices $\left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)^2$ and $\left(\begin{smallmatrix} p' & q' \\ r' & s' \end{smallmatrix}\right)^2$ are similar (see Lemma 4.1).

Observe that $\bar{M}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \approx T^2 \times S^1$. All PL involutions on $T^2 \times S^1$ are known (see K. Kwun and J. Tollefson [8]). Theorem C provides a complete classification of the orientation-reversing PL involutions on $M(0)$ (up to conjugation). Note that a space $\bar{M}\left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)$ is not homeomorphic to $M(b)$ for any b (see Lemma 4.3).

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