REPRESENTATION OF FUNCTIONS OF SEVERAL VARIABLES DEFINED ON THE TORUS

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Dedicated to Professor Casper Goffman on the occasion of his retirement at Purdue University.

1. Introduction. In answering a question, posed by Lusin, in connection with the representation of measurable functions, Men'šov [2, 8] proved that for any finite almost everywhere (abbreviated a.e.) measurable function f on $T=[0,2\pi]$, there exists a trigonometric series convergent to f a.e.. Bary [2] pointed out such a trigonometric series can be obtained from the term-by-term differentiation of the Fourier series of a primitive for the function f.

The representation of measurable functions of two variables by double trigonometric series was first studied by Dzhavarsheishvili [6]. Subsequently various representation problems for functions of several variables by multiple series were discussed by Dzagnidze ([4] and [5]) and Topuriya [9]. But their results are far from pointwise convergent representation. In connection with the Men'šov and Bary result for functions of several variables, it is natural to ask whether every finite a.e. measurable function f on T^n can be represented by an n-fold trigonometric series convergent to f a.e., summed either by squares or by rectangles [10].

In our previous work [3], we proved that any finite a.e. measurable function f on T^n can be represented by an n-fold trigonometric series convergent to $f(x)$ a.e. summed by squares. In the present article we show that any such a function can be represented by an n-fold trigonometric series convergent to the given function a.e. summed by rectangles.

Even for the case of a single variable, this result is quite deep since even for an integrable function the Fourier series may fail at every point to converge to the function. The result is much deeper in the case of functions of several variables because of the existence of functions continuous on T^n such that the rectangular partial sums of their n-tuple Fourier series diverge everywhere [7]. Neither of the proofs given by Men'šov and Bary can be extended to functions of several variables. We need to develop some fundamental tools. Several of the basic ideas for our main results are taken from our previous work [3]. For convenience and for notational simplicity, we give the proof of our theorem explicitly for functions on T^2.

2. Preliminaries and notations. By the 2-dimensional torus we mean the set of points $x=(x_1,x_2)$ from $T^2=[0,2\pi]\times[0,2\pi]$. Let $m=(m_1,m_2)$ be an integer lattice point of $\mathbf{R}^2$. Then for an integrable function F on T^2 the Fourier series for F is $S[x;F]=\Sigma_m \hat{F}_m e^{im\cdot x}$, where $\hat{F}_m=1/(2\pi)^2\int_{T^2}F(x)e^{-im\cdot x}dx$ with $m\cdot x=m_1x_1+m_2x_2$ and $dx=dx_1\,dx_2$.

To each Fourier series $S[x;F]$, there corresponds a trigonometric series $S'[x;F]=-\Sigma_m(m_1m_2\hat{F}_m e^{im\cdot x})$ obtained by term-by-term mixed differentiation of $S[x;F]$.

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