REALIZATIONS OF MAPS

R. Daniel Mauldin and A. H. Stone

1. INTRODUCTION

Let $(X,\mathscr{M},\lambda)$ and $(Y,\mathscr{A},\mu)$ be σ-finite measure spaces. For the results of this paper we can reduce our considerations to the case in which λ and μ are finite, and we assume that this reduction has been made. Throughout Sections 1–3 (but not in Section 4) we assume that μ is complete.

Let $\mathscr{E}$ be the measure algebra of $(Y,\mathscr{A},\mu)$, provided with the metric ρ given by $\rho([A],[B]) = \mu(A \triangle B)$, where $[A]$ is the measure class of A. We consider three types of maps $\phi: X \to \mathscr{E}$. First, we say that ϕ is "realizable" if there exists a measurable subset E of $X \times Y$ such that, for all $x \in X$, the section

$$E_x = \{y \in Y : (x,y) \in E\}$$

is in the measure class $\phi(x)$; such a set E will be called a "realization" of ϕ. Here the measure on $X \times Y$ is the completed product measure $\lambda \times \mu$. Second, we say that ϕ is "almost realizable" if there is a measurable $E \subset X \times Y$ such that $E_x \in \phi(x)$ for λ-almost all x. Finally, ϕ is "measurable" provided $\phi^{-1}(U)$ is λ-measurable for all open subsets U of the metric space $(\mathscr{E},\rho)$.

This paper arose from a problem raised (orally) by D. Maharam: Characterize the realizable functions ϕ. We give such a characterization in Theorem 3, Corollary, below. In particular, if λ is complete and $\mathscr{E}$ is separable, ϕ is realizable if and only if it is measurable. We also investigate some generalizations and sharpenings of this result.

We remark that, for each measurable $E \subset X \times Y$, E_x is μ-measurable for λ-almost all x. While this is not directly relevant to the question considered here, it does show that the question is a natural one.

We are grateful to the referee for suggestions that have improved the presentation of the results, and in particular for formulating Theorem 3, Corollary.

2. A CHARACTERIZATION OF REALIZABILITY

Let $\mathscr{F}$ be the family of all unions of finitely many "rectangles" $M \times A$, $M \in \mathscr{M}$, $A \in \mathscr{A}$, and let $\mathscr{R}$ be the σ-field generated by $\mathscr{F}$. We shall need the basic measure-theoretic facts summarized in the following theorem.

Received December 9, 1980. Revision received February 18, 1981.
The first author's research was supported by NSF Grant MCS-79-03784.
Michigan Math. J. 28 (1981).