THE ENERGY 1 METRIC ON HARMONICALLY IMMERSED SURFACES

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1. INTRODUCTION

In this paper we show that two theorems about minimally immersed surfaces are just special cases of more general statements about harmonically immersed surfaces. Both results characterize the harmonic immersion of a surface in the sphere in terms of the behavior of an immersion into the containing Euclidean space.

At the heart of these extensions is use of the energy 1 metric Γ, which at many points plays the role which the induced metric does on a minimal surface. We highlight the metric Γ here since it appears to be of general use in studying harmonically mapped surfaces. (See [7] and [9].)

Suppose a Riemannian metric $g = g_{ij}dx^i dx^j$ is specified on a surface S which is immersed in a manifold M^n with Riemannian metric $G = G_{\alpha\beta}du^\alpha du^\beta$. (We assume C^∞ smoothness throughout.) Let $\mathrm{I} = h_{ij}dx^i dx^j$ be the metric induced on S by G. The energy function e of the immersion is given by

$$e = e(g,\mathrm{I}) = \frac{g_{11}h_{22} + g_{22}h_{11} - 2g_{12}h_{12}}{2(g_{11}g_{22} - g_{12}^2)} = \frac{1}{2}\, tr_g \mathrm{I}. \tag{1}$$

Among all metrics on S proportional to g, only the choice $\Gamma = e(g,\mathrm{I})g$ on S yields energy function 1. Thus, we refer to Γ as the *energy 1 metric* of the immersion. Note that only the conformal class of g matters in the determination of Γ.

An immersion $X:(S,g) \to (M^n,G)$ is called *harmonic* in case X is extremal for the integral of $e(g,\mathrm{I})$ with respect to g. The Euler-Lagrange equation which must be satisfied is

$$\Delta_g X^\alpha + \overset{G}{\Gamma}{}_{\beta\gamma}^{\alpha} \frac{\partial X^\beta}{\partial x^i}\frac{\partial X^\gamma}{\partial x^j} g^{ij} = 0. \tag{2}$$

Here Δ_g is the Laplace-Beltrami operator of g, $\overset{G}{\Gamma}{}_{\beta\gamma}^{\alpha}$ is the Christoffel symbol associated with G, and $X = (X^\alpha)$. One sums on i, j, β, γ with $i, j = 1, 2$ and $\alpha, \beta, \gamma = 1, 2, \ldots, n$. Again, only the conformal class of g matters. In particular, (2) holds if and only if $X(S,\Gamma) \to (M^n,G)$ is harmonic for $\Gamma = e(g,\mathrm{I})g$. (See [2].)

An immersion $X:(S,g) \to (M^n,G)$ is *minimal* if and only if it is harmonic with $g \alpha \mathrm{I}$. For a minimal immersion, the energy 1 metric $\Gamma = e(g,\mathrm{I})g$ is I itself. It is

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