ON FUCHSIAN GROUPS OF DIVERGENCE TYPE

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1. INTRODUCTION

Let Γ be a Fuchsian group in the unit disk $\mathbf{D}$ with identity ι. We assume throughout that 0 is not an elliptic fixed point and denote by ρ the radius of the maximal disk $\{|z| < \rho\}$ that does not contain Γ-equivalent points. We also assume that Γ does not have a compact fundamental domain, so that $\mathbf{D}/\Gamma$ is an open Riemann surface.

If Γ is of convergence type, that is if $\Sigma_{\gamma\in\Gamma}(1 - |\gamma(0)|) < \infty$, then the Blaschke product

$$g(z) = z \prod_{\gamma\in\Gamma,\gamma\neq\iota} \frac{|\gamma(0)|}{\gamma(0)}\gamma(z) \qquad (z \in \mathbf{D}) \tag{1.1}$$

is called the *Green's function* of Γ with respect to 0; the positive harmonic function $-\log|g(z)|$ corresponds to the Green's function on $\mathbf{D}/\Gamma$.

Let now Γ be of divergence type; in the terminology of classification theory this means that $\mathbf{D}/\Gamma \in O_G$. We consider analogues of Green's functions; their harmonic counterparts on $\mathbf{D}/\Gamma$ are, for instance, the Evans function [15, p. 350] and Tsuji's modified Green's function [16, p. 455].

Let $\mathfrak{G}(\Gamma)$ denote the class of all functions $f(z) = z + a_2z^2 + \dots$ analytic in $\mathbf{D}$ with

$$|f(\gamma(z))| = |f(z)| \qquad \text{for} \qquad \gamma \in \Gamma, \quad z \in \mathbf{D} \tag{1.2}$$

such that $|f(z)|$ is bounded away from 0 in $\mathbf{D}\setminus \bigcup_{\gamma\in\Gamma} \gamma(D_0)$ for a suitable disk D_0 around 0.

We shall show that the bounds in the last condition are actually only dependent on ρ (see Theorem 3). Every function $f \in \mathfrak{G}(\Gamma)$ is normal, and there is a natural fundamental domain associated with it (see Theorem 2). Perhaps the main result (Theorem 4) is that the functions $f \in \mathfrak{G}(\Gamma)$ that remain bounded at the parabolic vertices satisfy the best possible estimate

$$\log^+|f(z)| = o\left(\frac{1}{1-|z|}\right) \qquad \text{as} \qquad |z| \to 1 - 0. \tag{1.3}$$

Let $L_0(\Gamma)$ denote the set of all $\zeta \in \partial\mathbf{D}$ for which there exist $\gamma_k \in \Gamma$ with

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