FINITENESS AND RIGIDITY THEOREMS FOR HOLOMORPHIC MAPPINGS

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In this paper we study the complex spaces $\mathrm{Hol}_k(X,Y)$ of holomorphic maps of rank $\geq k$ from a compact complex space X into a complex manifold Y. Our results are of the following type: If Y satisfies certain conditions, then for particular k, the space $\mathrm{Hol}_k(X,Y)$ is either discrete or finite, independent of X. Of particular interest is the case $k = \dim Y$, where $\mathrm{Hol}_k(X,Y)$ is the space of surjective holomorphic maps.

Our results are modelled on the classical result of de Franchis that for Y a compact Riemann surface of genus greater than 1, the number of surjective holomorphic maps is finite. Lang [9] raised the question whether finiteness holds for Y compact hyperbolic. Kobayashi and Ochiai [8] proved that the set of surjective meromorphic maps from a Moisĕzon space into a compact complex space of general type is finite. Recently Noguchi and Sunada [13] proved that if X is Moisĕzon and $\Lambda^k T_Y$ is Grauert negative, then the number of meromorphic maps of rank $\geq k$ from X to Y is finite. Borel and Narasimhan [1] have also proved discreteness results for holomorphic maps. Similar finiteness theorems for harmonic mappings are given by Lemaire [10].

Our results, which are valid only for holomorphic maps, complement the results of [8] and [13] mentioned above. Theorem 1 says that $\mathrm{Hol}_{k+1}(X,Y)$ is discrete if the holomorphic tangent bundle T_Y satisfies a k-pseudo-convexity condition. In Theorem 1, Y may be noncompact. A consequence of this result (Corollary 2) is that if Y is a compact hermitian manifold with negative holomorphic sectional curvature, then the set of surjective holomorphic maps is finite. We also prove that if Y is an n-dimensional compact Kähler manifold with $c_1(Y)$ represented by a negative semidefinite form and either $c_n(Y) \neq 0$ (Theorem 1) or $c_1^n(Y) \neq 0$ (Theorem 3), then $\mathrm{Hol}_n(X,Y)$ is discrete. In particular, if Y is compact Kähler with first Chern class zero and Euler class nonzero (for example, a Kähler K3 surface), then the space of surjective holomorphic maps onto Y is discrete.

Our method of proof is to consider a one-parameter family of holomorphic maps and to view the derivative with respect to the deformation parameter as a holomorphic mapping from X into the tangent bundle of Y. This method was used independently by Urata [14] to prove Corollary 2.

In the following we let X be a compact, connected complex space, and we let Y be a connected n-dimensional complex manifold. We denote by $\mathrm{Hol}(X,Y)$ the space of holomorphic maps from X to Y, equipped with the compact-open topology. By a well-known result of Douady, $\mathrm{Hol}(X,Y)$ is a complex space (see Lemma 3). If $f \in \mathrm{Hol}(X,Y)$ we define rank f to be the maximum rank of f on the regular points of X; thus $f(X)$ is an analytic space and $\dim f(X) = \operatorname{rank} f$. We let

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