SOME ORDERINGS INDUCED BY SPACES OF ANALYTIC FUNCTIONS

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We consider here some orderings on subsets of the complex plane (or the Riemann sphere) that are induced by certain spaces of analytic (or meromorphic) functions, and write $E \leq_S E'$, where S is the space of functions under consideration. The problem that arises is to give geometrical conditions on E and E' that $E \leq_S E'$. We give complete solutions when $S = N$, the space of normal meromorphic functions in the open unit disc $\mathbf{D}$, and when $S = H^\infty(\mathbf{D})$, the space of bounded analytic functions in $\mathbf{D}$. We briefly discuss the problem when $S = B$, the Bloch space, and when $S = H^p(\mathbf{D})$, $0 < p < \infty$, the Hardy space in $\mathbf{D}$. These problems have a family resemblance, but are different in detail.

Definition N. Let E and E' be subsets of the extended complex plane $\mathbf{C}^\wedge$. Suppose that for every meromorphic function f in $\mathbf{D}$,

$$[\sup\{f^\#(z)(1-|z|^2) : f(z) \in E'\} < \infty] \Rightarrow [\sup\{f^\#(z)(1-|z|^2) : f(z) \in E\} < \infty]$$

We then say $E \leq_{\mathrm{N}} E'$.

Problem N. (Solved in this paper.) Find geometrical necessary and sufficient conditions on E and E' that $E \leq_{\mathrm{N}} E'$.

Remark N. We emphasize that in this and our other order definitions, z runs over $f^{-1}(E')$ and $f^{-1}(E)$ respectively, and not z runs over E' amd E respectively, which would give the problems an entirely different flavor. In [6], it was proved (in other language) that if card $E' \geq 5$, then $E \leq_{\mathrm{N}} E'$ for $E = \mathbf{C}^\wedge$ and hence for every set E. The letter N is used here for "Normal"—f is normal [1] if and only if $\sup\{f^\#(z)(1-|z|^2) : z \in \mathbf{D}\} < \infty$. where $f^\#$ is the spherical derivative

$$f^\#(z) = \frac{|f'(z)|}{1 + |f(z)|^2}.$$

Our main theorem gives necessary and sufficient geometrical conditions that $E \leq_{\mathrm{N}} E'$.

Definition B. Let E and E' be subsets of the complex plane $\mathbf{C}$. Suppose that for every analytic function f in $\mathbf{D}$,

$$[\sup\{|f'(z)|(1-|z|^2) : f(z) \in E'\} < \infty] \Rightarrow [\sup\{|f'(z)|(1-|z|^2) : f(z) \in E\} < \infty].$$

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