STABILIZATIONS OF PERIODIC MAPS ON MANIFOLDS

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1. INTRODUCTION

In this paper we will study the graded unrestricted unoriented cobordism ring of involutions, $I_*(\mathbf{Z}_2)$, and an endomorphism, Γ, of $I_*(\mathbf{Z}_2)$ of degree $+1$. We use this endomorphism to define ideals in $I_*(\mathbf{Z}_2)$ by

$$\mathscr{A}_n = \{x + \Gamma^n(x) : x \in I_*(\mathbf{Z}_2) \quad \text{and} \quad \varepsilon(\Gamma^j(x) = 0 \quad \text{for} \quad 0 \leq j < n\}.$$

The ideal $\mathscr{A}_1$ plays a more important part in our theory than the remainder of these ideals. We prove that $I_*(\mathbf{Z}_2)/\mathscr{A}_1$ is a polynomial ring over MO_* and over $\mathbf{Z}_2$.

We apply this result about $I_*(\mathbf{Z}_2)/\mathscr{A}_1 \equiv \Lambda(\mathbf{Z}_2)$ to prove Boardman's five-halves theorem. After noting that $\Lambda(\mathbf{Z}_2) \equiv MO_*(BO)$, we use the results of [2] to exhibit explicit polynomial generators for $\Lambda(\mathbf{Z}_2)$ whose underlying manifolds generate MO_* as a polynomial ring over $\mathbf{Z}_2$. We then consider two filtrations on $\Lambda(\mathbf{Z}_2)$. After seeing how these filtrations behave on the polynomial generators for $\Lambda(\mathbf{Z}_2)$ and on polynomials in these generators, the Five-halves theorem and its converse follow. We then look at an application of this theorem and its method of proof to a conjecture about flat manifolds.

We notice that certain elements in the ideals $\mathscr{A}_n$ behave much like the polynomials in $1 + t^n$ in $\mathbf{Z}_2[t]$. Knowing how to factor the cyclotomic polynomials and, hence, $1 + t^n$ in $\mathbf{Z}_2[t]$, we are led to a factorization of the elements $x^k + \Gamma^n(x^k)$ in $\mathscr{A}_n$ in an analogous manner.

2. PRELIMINARY MATERIAL

We will use $I_*(\mathbf{Z}_2)$ to denote the graded unrestricted unoriented cobordism ring of smooth manifolds with involution; $MO_*(\mathbf{Z}_2)$, the graded unoriented cobordism ring of smooth manifolds with fixed point free involutions; MO_*, the graded unoriented Thom cobordism algebra; and $\mathscr{M}_*$, the graded unoriented cobordism ring of principal $O(k)$ bundles with $\mathscr{M}_n = \sum_{j=0}^{n} MO_j(BO(n-j))$ and $MO_n(BO(O)) = MO_n$, by definition.

Conner and Floyd completely determined the additive structure of $I_*(\mathbf{Z}_2)$ in the following theorem.

Received October 23, 1978. Revision received June 1, 1979.
This work formed a part of the author's dissertation at LSU.

Michigan Math. J. 27 (1980).