THREE-HOLED SPHERES AND RIEMANN SURFACES

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INTRODUCTION

Let R be a compact Riemann surface of genus g bigger than one. A Jenkins-Strebel differential φ on R is a holomorphic, quadratic differential on R all of whose noncritical, horizontal trajectories are closed. Such a differential gives a natural way of decomposing R into annuli whose boundaries consist of the critical, horizontal trajectories of φ.

In this article two procedures are given for constructing analogous holomorphic, quadratic differentials on R which are associated with a decomposition of the surface into three-holed spheres. In one case, it turns out that one again obtains Jenkins-Strebel differentials. In a second case, the form of the differentials so constructed is not known.

The first section summarizes certain facts about uniformization of three-holed spheres. Such domains can be uniformized by deleting three intervals from the real axis and there are simple inequalities for expressing compactness in terms of the endpoints of these intervals. The compactness condition is needed for a normal families argument used in section 3. That the uniformization can be achieved by removing three intervals was already observed by Jenkins in [8].

In the second section, variational formulas for certain natural functions on $T^{\#}(R)$, the reduced Teichmüller space of a Riemann surface R, are derived.

In the third section, these variational formulas are used to construct global quadratic differentials on a surface of genus g naturally associated with a partition of that surface into $2g-2$ three-holed spheres.

1. UNIFORMIZATION OF THREE-HOLED SPHERES

Let R be a Riemann surface of finite type. This means it can be obtained from a compact surface by deleting a finite number of continua. The reduced Teichmüller space of R, $T^{\#}(R)$, is defined in [6] and so is the space $Q(R)$ of holomorphic, quadratic differentials on R which are real with respect to boundary uniformizers along the boundary of R. Let g be the genus of R, m be the number of deleted continua each of which contains more than one point, and n be the number of deleted continua each of which consists of exactly one point. (m is the number of "holes" and n is the number of "punctures.") Let ρ be the real

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