TORUS KNOTS IN THE COMPLEMENTS OF LINKS AND SURFACES

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Our major result is Theorem 2, which states that if $S \subset S^3$ is any nonsingular, nonseparating orientable surface (possibly disconnected), then there exists a torus knot t with mirror image t' such that $(t \# t') \cap S = \emptyset$ and $S \subset F$, where F is a minimal spanning surface (fiber) for $t \# t'$. The basic construction gives insight into the structure of closed 3-manifolds, and some important known results follow as corollaries.

The notation and techniques are standard (cf. [3], [6], [10]). We work exclusively in the PL category. The symbols $\partial(\ldots)$, $N(\ldots)$, and $CL(\ldots)$ denote, respectively, the boundary, regular neighborhood, and closure of the object $(\ldots)$. We shall have several occasions to move links and surfaces by ambient isotopies, making the implicit assumption that each may be reversed when the construction is complete.

We begin by constructing torus knots in the complements of links.

THEOREM 1. *Let $k \subset S^3$ be a tame link. There exists a torus knot $t \subset S^3$ such that $t \cap k = \emptyset$ and $k \subset F$, where F is a minimal spanning surface for t.*

Proof. We say that a link in R^3 is in *square bridge position* with respect to the plane $z = 0$ if the projection onto the plane is regular and if each segment above the plane projects to a horizontal segment and each one below to a vertical segment. Our link $k \subset S^3$ may be represented as a closed braid [1, p. 42], and we may assume k lies in a 3-cell which has been identified with $[0,1] \times [0,1] \times [-1,1] \subset R^3$. It is now a simple matter to make each overcrossing horizontal and each undercrossing vertical, and an additional ambient isotopy (cf. figure 1) will put k into square bridge position with respect to the "plane" $[0,1] \times [0,1] \times \{0\}$. (Note that the minimum bridge number of all such square bridge representations is an integral link invariant; we call it the *square bridge number.*)

Let $T \subset S^3$ be a torus which determines the genus one Heegard Splitting (U, V) of S^3. We may assume $T \cap ([0,1] \times [0,1] \times [-1,1]) = [0,1] \times [0,1] \times \{0\}$ and $[0,1] \times [0,1] \times [0,1] \subset U$. Each disk $[0,1] \times \{b\} \times [0,1]$ thus lies in a properly embedded, nonseparating disk $D \subset U$, so $k \cap U \subset \bigcup_{i=1}^{r} D_i$. Similarly, $k \cap V \subset \bigcup_{i=1}^{s} E_i$, where each $E_i \subset V$ is a properly embedded, nonseparating disk containing $\{a\} \times [0,1] \times [-1,0]$ and each $D_i \cap E_j$ is a singleton. If $(r,s) \neq 1$, add more disks D, with $D \cap k = \emptyset$, until $(r,s) = 1$. The curves ∂D_i and ∂E_i inherit an orientation from the coordinatization of $[0,1] \times [0,1] \times \{0\}$.

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