ON ROOTS IN FREE GROUPS

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INTRODUCTION

Let R and S be disjoint subsets of a free group F such that $R \cup S$ is linearly independent modulo F'; it is shown that the normal closure of R in F intersects $gp(S)$ trivially—this is the Proposition of Section 2. Since S is independent modulo F', the subgroup generated by it is freely generated by it; thus we obtain

THEOREM 1. *Let R and S be disjoint subsets of a free group F such that $R \cup S$ is linearly independent modulo F'. In the group presented on F with defining relators R, the image of S freely generates a free group.*

This theorem is closely related to two theorems of Magnus that are concerned with presentations whose defining relators form part of a basis modulo the derived group: It generalizes his theorem which states that if a group with $n + r$ generators and r defining relators can be generated by n elements, then it is freely generated by them, and is related to his theorem which states that if G is a group with $n + r$ generators and r defining relators and G/G' is free abelian of rank n, then the generators of G may be chosen so that n of them freely generate a free group [7].

The proposition of Section 2 is applied here to the problem of finding the roots of an element in a free group. Let a and b be elements of a group G. If a is in the normal closure in G of b, then b is said to be a root of a in G, and we write $b \underset{G}{\rightarrow} a$. In 1930 [5] Wilhelm Magnus posed the problem of finding all the roots of a given element of a free group F. For F free with a basis $x, y, z, w, \ldots$ he found all roots of x, $[x,y]$, and x^2y^p for p a prime, as well as of certain other elements. However, for example, the set of all roots of x^ky^k, for k not a prime, is not known. Nor is it known if the problem of finding all the roots of an arbitrary element of a free group is solvable. In view of the difficulty of finding all the roots of a given element in a free group, the following theorem of Arthur Steinberg [On equations in free groups. Michigan Math. J. 18(1971), 87–95] is very powerful; it can, for example, be combined with the results of Magnus to ascertain all the roots of $[[x,y], [z,w]]$.

THEOREM 2. *Let F be the free group with basis $x_1, \ldots, x_n, y_1, \ldots, y_m, \ldots, z_1, \ldots, z_\ell$. Let*

$$x = X(x_1, \ldots, x_n), y = Y(y_1, \ldots, y_m), \ldots, z = Z(z_1, \ldots, z_\ell)$$

be non-trivial elements, none of which is a proper power in F. Let E be the subgroup of F that has basis $x, y, \ldots, z$. Let

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Received February 10, 1978. Revision received April 10, 1979.

Michigan Math J. 27 (1980).