EXTENSION OF A THEOREM OF SZEGÖ

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We extend to more general measures a theorem due to Szegö [7] which states that the boundary values of a (nonidentically zero-valued) function in classical Hardy space [4] are log integrable with respect to normalized Lebesgue measure on the unit circle. Our interest in this area arose from a desire to understand the interplay between the existence of bounded point evaluations for a measure and the parts of the measure carried by the open unit disc and the boundary of the unit disc.

Let D denote the open unit disc of the complex plane, $\mathbf{C}$. In this paper all measures considered will be finite positive compactly supported Borel measures carried by $\bar{D}$. For a measure μ let $H^2(\mu)$ denote the closure in $L^2(\mu)$ of the set of polynomials in z. If for some $\lambda \in \mathbf{C}$ the point evaluation functional $p \to p(\lambda)$ defined on polynomials p is bounded, i.e.,

$$\sup\{|p(\lambda)|/\|p\|_\mu : p \text{ is a nonzero polynomial}\} < \infty,$$

then we say that μ has a *bounded point evaluation* at λ or a *b.p.e.* at λ for short.

Let K be a compact set. Then K contains *an exposed arc* J if there exists a simply connected open set E such that $E \cap K = J$ and J is the range of a smooth Jordan curve. A bounded component of K is called a *hole* of K.

For a measure μ carried by $\bar{D}$ for which

(*) there is a hole H of the support of μ so that $\bar{H}$ has an exposed arc Γ with $\Gamma \subset \partial D$

we say that μ satisfies (*) (with respect to H and Γ). Denote the open subarc obtained from Γ by the removal of the endpoints by Γ^0 (if $\Gamma = \partial D$ then let $\Gamma^0 = \partial D$). For example, $d\sigma$, normalized Lebesgue measure on ∂D, satisfies (*) (with respect to D and ∂D).

The *filling in holes* theorem due to Bram [2] interpreted in our context states that for a measure μ satisfying (*) either

(1) μ has a *b.p.e.* at every $\lambda \in H$

or else

(2) μ has no *b.p.e.*'s in H.

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