TURÁN'S SECOND THEOREM ON SUMS OF POWERS OF COMPLEX NUMBERS

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Let $z_1, \ldots, z_n, b_1, \ldots, b_n$ be complex numbers such that $1 = |z_1| \geq |z_2| \geq \ldots \geq |z_n|$ and define $S_k = b_1 z_1^k + \ldots + b_n z_n^k$. P. Turán [3] considered the problem of finding a lower bound for

$$M_{m,n} = \min \max_{m+1 \leq k \leq m+n} |S_k|,$$

where the min is taken over all possible values of $z_1, \ldots, z_n$ subject to the above constraints. He proved in [3] that

$$M_{m,n} \geq \left(\frac{n}{24e^2 (m + 2n)} \right)^n \min_{1 \leq j \leq n} |b_1 + \ldots + b_j|$$

and applied this result to various problems, including the question of the distribution of the zeros of $\zeta(s)$ in the critical strip.

Later V. T. Sos and P. Turán [2] improved the estimate by showing that

$$(1) \qquad M_{m,n} \geq \left(\frac{n}{A(m + n)} \right)^n \min_{1 \leq j \leq n} |b_1 + \ldots + b_j|$$

holds with $A = 2e^{1+4/e}$. It was pointed out by Uchiyama [4] that the method of [2] will actually give (1) with the better constant $A = 8e$. In fact, it is not hard to see that using the same method one can get

$$M_{m,n} \geq \left(\frac{m}{m + n} \right)^m \left(\frac{n}{8(m + n)} \right)^n \min_{1 \leq j \leq n} |b_1 + \ldots + b_j|;$$

here the factor $(m/(m+n))^m$ always exceeds e^{-n} but tends to e^{-n} as $m \to \infty$.

In this paper we give a further improvement of the constant A in (1); our result is $A \leq 7.81e$. At the cost of some complications, our method could undoubtedly be modified to give a slightly smaller constant.

The problem of finding a lower bound for the best possible constant A in (1) has been considered. The best known result is $A \geq 4e$, due to Makai [1].

We need the following lemma in our proofs.

LEMMA. *Let m be a positive integer and let $z_1, \ldots, z_n$ be any complex numbers.*

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